

Multiple zeta functions and double wrapping
in planar $\mathcal{N} = 4$ SYMSébastien Leurent^a, Dmytro Volin^{b,c}^a*Theoretical Physics group, Imperial College, South Kensington Campus,
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Abstract

Using the FiNLIE solution of the AdS/CFT Y-system, we compute the anomalous dimension of the Konishi operator in planar $\mathcal{N}=4$ SYM up to eight loops, i.e. up to the leading double wrapping order. At this order a non-reducible Euler-Zagier sum, $\zeta_{1,2,8}$, appears for the first time. We find that at all orders in perturbation, every spectral-dependent quantity of the Y-system is expressed through multiple Hurwitz zeta functions, hence we provide a *Mathematica* package to manipulate these functions, including the particular case of Euler-Zagier sums. Furthermore, we conjecture that only Euler-Zagier sums can appear in the answer for the anomalous dimension at any order in perturbation theory.

We also resum the leading transcendentality terms of the anomalous dimension at all orders, obtaining a simple result in terms of Bessel functions. Finally, we demonstrate that exact Bethe equations should be related to an absence of poles condition that becomes especially nontrivial at double wrapping.

Keywords:

Y-system, FiNLIE, integrability, perturbative quantum field theory, AdS/CFT correspondence

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Interactive feature: If you are reading this article as a pdf file using a viewer that supports JavaScript (like Adobe Reader), you can click a sharp symbol, for instance this one $\#$, whenever you encounter it in the text. When such symbol is clicked, a pop up window with a *Mathematica* code or example relevant to the context will appear. The *Mathematica* code requires a number of packages from [1] $\#$. More examples and further explanations how to use packages are given in the notebook `usage.nb` in [1].

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1. Introduction

During the last decade, there was a remarkable progress in applying the AdS/CFT integrability to solve the planar $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM), see [2] for a review. In this context, one of the best studied directions is the AdS/CFT spectral problem – the computation of anomalous dimensions of gauge invariant local operators or, equivalently, of energies of dual string configurations. One typically demonstrates how to solve this problem by considering some particular operator or a class of operators; the most known example are the so called twist J spin S states. The shortest member of this family whose anomalous dimension γ is not protected by supersymmetry is $\text{Tr} Z D^2 Z$, where D is a light-cone covariant derivative and Z is a complex scalar field of $\mathcal{N}=4$ SYM. This state known as the Konishi operator corresponds to $J=2$ and $S=2$. The Konishi operator is interesting both from the point of view of $\mathcal{N}=4$ SYM and its string dual: on the gauge side it appears among the leading terms of operator product expansions, and on the string side it is among excitations with the lowest energy. From the point of view of integrability, the anomalous dimension of the Konishi operator is specifically hard to compute perturbatively in the sense that the so called wrapping corrections [3] start to contribute earlier than for the anomalous dimensions of other states. Explicitly, the leading wrapping correction appears at four loops in the perturbative expansion in the 't Hooft coupling constant $g^2 = \frac{g_{YM}^2 N_c}{16\pi^2}$. Whereas the anomalous dimensions of “long” operators which are free of wrapping corrections (e.g. the ones with $J \rightarrow \infty$) can be studied by solving the algebraic equations of the Beisert-Staudacher asymptotic Bethe Ansatz [4, 5], the presence of wrapping corrections requires to solve functional equations instead: the Gromov-Kazakov-Vieira Y-system [6].

Over the past few years, there has been quick progress in the understanding of how to compute anomalous dimensions of “short” operators (the ones that receive wrapping corrections). One already has reasonably precise numerical values for anomalous dimensions of the Konishi state [7, 8] and of various twist J spin S operators [9]. At strong coupling, these dimensions were found analytically at two [10] and three [11] loops, with the two-loop result matching existing computations [12, 13, 14] from the string theory, whereas at weak coupling the analytical answer for $J = 2$ case was found up to five loops: the results [15, 16, 17, 18] for $S = J = 2$ case coincide with perturbative quantum field theory computations that reached four [19, 20, 21]

and five loop [22] orders, while the analytical continuation of the arbitrary S result [23, 24, 25] to $S = -1$ agrees with the prediction from the BFKL equation [26]. The Konishi state was recently analysed even in greater detail and by now the six [27] and seven loop [28] results for its anomalous dimension are available. There are also improvements in analysing other operators than the twist J spin S operators [29, 30], in studying twisted [31, 32, 33, 34, 35] and q -deformed [36, 37] versions of the AdS/CFT spectral problem, and in computing angle-dependent cusp anomalous dimensions and related quantities in $\mathcal{N}=4$ SYM from boundary thermodynamic Bethe Ansatz [38, 39, 40].

Though all these advancements look encouraging, a worrisome sign is that a number of important results was obtained by approaches that are stiff for improvement and generalizations.

At strong coupling, using the method of [10] seems technically infeasible beyond two loops; for instance, the three-loops result [11] was obtained using two-loop findings and an extra knowledge about the Basso's slope [41]. This situation is somehow similar to the computation of the cusp anomalous dimension by taking a certain limit of the generalized scaling function [42, 43, 44]. While this approach becomes a burden beyond two loops, the cusp anomalous dimension can be computed by other means [45, 46] to any desired order, and we might hope to find such means for the dimensions of short operators as well.

At weak coupling, the analytic five-loop result in [18] was obtained by a perturbative solution of the thermodynamic Bethe Ansatz equations, and there are conceptual obstacles in generalizing the method of [18] to higher loops [47]. Another approach of [15, 16] is to use Luscher formulae. It is an efficient way to account for the so called single wrapping effects. However, the Luscher formulae can be derived only for the vacuum state, while their generalizations to excited states (in particular, to the Konishi state) is a conjecture. At double wrapping orders, Luscher formulae are known and were used for the vacuum state of the γ -deformed theory [34], however it is not clear how to proceed and to generalize them to excited states [48]. Hence, there is currently a theoretical bound for applying Luscher formulae, which is seven loops for the case of the Konishi anomalous dimension and which was reached in [28].

The limitations of these methods and results indicate that some fundamental properties of the AdS/CFT integrable system are still not understood or were not used in these computations. In particular, the above-mentioned five-loop result at weak coupling only partially simplifies the orig-

inal way to find the spectrum from the thermodynamic Bethe Ansatz equations [49, 50, 51], whereas, in later developments, more simpler structures were found behind them: these equations were shown to be derivable from the Gromov-Kazakov-Vieira Y-system (or equivalently the Hirota T-system on the T-hook), if one supplements it with certain constraints [52, 53] on discontinuities of the Y-functions. Then, these constraints were reduced in [54] to simple group-theoretical conditions on the T-functions.

The T-functions obey the Hirota equation, which is a generalization of character identities where a dependence on a spectral parameter is added. The Hirota equation can be solved by the so called Wronskian solution, a generalization of the Weyl character formula, which allows one to parameterize the T-functions in terms of a finite set of Q-functions. These properties were for a long time known for various integrable models [55], and their equivalents were discovered for the case of the AdS/CFT integrability in [56, 57], see also [58, 59, 60] and references therein for a more generic set up.

By using these “algebraic” findings, the Wronskian parameterization of the T-functions and the group-theoretical constraints on them, a finite set of nonlinear integral equations (FiNLIE) which allows one to compute the anomalous dimensions of several operators was derived in [54]. By contrast, the thermodynamic Bethe Ansatz equations form an infinite set of equations.

In this paper, following our previous work [27], we apply the FiNLIE, and hence the findings discussed above, to show that the weak coupling expansion can be carried to an arbitrary order in perturbation theory. We demonstrate this by an explicit computation of the Konishi anomalous dimension up to eight loops, a benchmark order at which the double wrapping effects become important for the first time.

The goal of this work was however not only to improve the methods for weak coupling expansion. Understanding of the AdS/CFT integrability is far from being ultimate. In particular, it is conceivable that, instead of the mirror thermodynamic Bethe Ansatz resulting in an unwieldy infinite set of equations, there should be a better way to derive the Hirota system and constraints on it. In our ongoing research [61] we are finding new interesting features behind the AdS/CFT T- and Y-systems. On the one hand, these features should yield a more efficient way to compute anomalous dimensions. On the other hand, we hope that they will shed additional light on the fundamental aspects of the AdS/CFT integrability which would also be important beyond the spectral problem. It appeared that having explicit analytical results is necessary to guideline our research, and that not all of the effects

manifest themselves at single wrapping orders. Hence we had to solve the AdS/CFT Y-system up to a double wrapping order, and we present such solution in this paper.

Our analysis of the perturbative weak coupling behaviour of the FiNLIE showed that all its quantities can be expressed through the so called multiple Hurwitz zeta functions. In the following section 2 we acquaint the reader with these functions and with basic operations on them. Section 3 describes the FiNLIE adjusted to weak coupling expansion and gives some simplified examples which show how to use it. All further technical details are available online [1] in the *Mathematica* notebook format. In section 4 we give our main result: the eight-loop Konishi anomalous dimension, and in section 5 we make some crosschecks by performing an expansion in the inverse powers of transcendentality of zeta functions. Finally, we summarize our results in the conclusions section, where we also discuss how the exact Bethe equations are related to the regularity conditions on a solution of the FiNLIE. The analyticity properties encoded into these exact Bethe equations are an example of a property which acquires new qualitative features at double wrapping.

To set up terminology, let us note that we define the anomalous dimension γ of an operator as $\gamma = \Delta - \Delta_0$, where Δ is the total conformal dimension and Δ_0 is the classical dimension. For the Konishi operator $\text{Tr } Z D^2 Z$ one has $\Delta_0 = 4$. The energy E of the string configuration dual to the Konishi state is defined with respect to the BMN vacuum $\text{Tr } Z^2$ so that $E = \gamma_{\text{Konishi}} + 2$. The Konishi operator is a member of the so called Konishi supersymmetric multiplet. Another member of this multiplet is $\sum_{i=1}^6 \text{Tr } \Phi_i^2$, which is an R -symmetry singlet formed from real scalar fields Φ_i of $\mathcal{N}=4$ SYM¹. As all the operators from the Konishi multiplet, this operator has $\gamma = \gamma_{\text{Konishi}}$, though its classical dimension is different: $\Delta_0 = 2$.

2. Multiple Hurwitz zeta functions

In section 3 we will show that all the quantities of the FiNLIE, and hence of the Y-system, are expressed at any given order of weak coupling expansion through multiple Hurwitz zeta functions. The goal of the present section is to define these functions and to list their essential properties. We start with a specific case of great importance: Euler-Zagier sums.

¹One defines Z as e.g. $Z = \Phi_1 + i\Phi_2$.

2.1. Euler-Zagier sums²

The Euler-Zagier sums, also known as multiple zeta values (MZV), are defined as follows

$$\zeta_{a_1, a_2, \dots, a_k} = \sum_{0 < n_1 < n_2 < \dots < n_k < \infty} \frac{1}{n_1^{a_1} n_2^{a_2} \dots n_k^{a_k}}. \quad (1)$$

$w = \sum_{i=1}^k a_i$ is called the weight, or transcendentality, of the sum, k is called the depth of the sum. The sum is convergent for $a_k > 1$, $a_k + a_{k-1} > 2$, \dots , $\sum_{i=1}^k a_i > k$. In the following we will define a regularization which allows us to define MZV in the marginally divergent case, i.e when strict inequalities $>$ above are weakened to \geq . We also assume that $a_i \geq 1$, otherwise the sum can be straightforwardly reduced to sums of lower depth. For instance:

$$\zeta_{-1, a} = \sum_{n_2=1}^{\infty} \sum_{n_1=1}^{n_2-1} \frac{n_1}{n_2^a} = \frac{1}{2} (\zeta_{a-2} - \zeta_{a-1}). \quad (2)$$

Stuffle algebra. MZVs form a ring over \mathbb{Q} . Indeed, consider for example

$$\zeta_a \zeta_b = \sum_{n, m} \frac{1}{n^a m^b} = \left(\sum_{n < m} + \sum_{n > m} + \sum_{n=m} \right) \frac{1}{n^a m^b} = \zeta_{a, b} + \zeta_{b, a} + \zeta_{a+b}. \quad (3)$$

Clearly, if we consider two arbitrary MZVs, we can repeat this logic to split the sum and express their product as a linear combination of other MZVs. The equalities that we obtain this way are known as stuffle relations.

Shuffle algebra. At the same time, there is another inequivalent way to express a product of MZVs through a linear combination of other MZVs. For this we use the Feynman representation of Euler-Zagier sums:

$$\zeta_{a_1+1, a_2+1, \dots, a_k+1} = \int_{\infty > t_k > \dots > t_1 > 0} \left(\prod_{i=1}^k \frac{dt_i}{a_i! (e^{t_i} - 1)} \right) (t_1 - t_2)^{a_1} \dots (t_{k-1} - t_k)^{a_{k-1}} t_k^{a_k}. \quad (4)$$

²For a more comprehensive discussion of this subject the reader may consult [62, 63, 64] and references therein.

In the stuffle case we had to split summation whereas here we will split domain of integration, e.g.³:

$$\zeta_1 \zeta_2 = \int_{t_1, t_2} d\mu t_1^0 t_2^1 = \int_{t_1 < t_2} d\mu t_2 + \int_{t_1 > t_2} d\mu ((t_2 - t_1) + t_1) = 2\zeta_{1,2} + \zeta_{2,1}, \quad (5)$$

where $d\mu$ is the measure of integration defined in (4).

Using this logic, we define the so called shuffle relations. This name comes from the fact that we shuffle the ordering of the integration variables in all possible ways. In the mathematical literature a different integral representation is preferred, the one used to define multiple polylogarithms [65], however the net result is the same.

Both the shuffle and stuffle products are implemented in our *Mathematica* package `zetafunctions.m` [1]. #

Combinations of shuffle and stuffle products allow one to generate non-trivial relations between MZVs. A classical example is that the relations

$$\begin{aligned} \zeta_1 \zeta_2 &= \zeta_{1,2} + \zeta_{2,1} + \zeta_3 && \text{(stuffle product)}, \\ \zeta_1 \zeta_2 &= 2\zeta_{1,2} + \zeta_{2,1} && \text{(shuffle product)} \end{aligned} \quad (6)$$

give the Euler relation

$$\zeta_{1,2} = \zeta_3. \quad (7)$$

Diophantine conjecture. It is conjectured [66] that all the algebraic relations between Euler-Zagier sums are generated by shuffle and stuffle relations. One immediate consequence of this conjecture is that MZVs of different weight are algebraically independent. This conjecture called the *diophantine conjecture* is not proven. However, one can test it by exploiting integer relation algorithms in experimental mathematics (see [67] and references therein). These algorithms allow performing an efficient and systematic search for relations of the type $\sum_I c_I \zeta_I = 0$, where the summation is over a set of multi-indices I and c_I are integers. In particular, these algorithms can show that no relation exists with coefficients c_I -s smaller than a certain magnitude. The diophantine conjecture found a solid support using this approach.

³In this example, both ζ_1 (in the l.h.s.) and $\zeta_{2,1}$ are marginally divergent. However, the formal manipulations that we write disregarding the convergency issue give a correct result if one uses the regularization prescription defined in Appendix A.1. In general, the shuffle algebra that we derive this way is valid for a product involving at most one marginally divergent MZV in this regularization prescription.

Based on the diophantine conjecture, one can show [66] that the number M_w of independent irreducible^{4,5} MZVs of weight w can be found from the following generating relation [68]: $1 - x^2 - x^3 = \prod_{w>0} (1 - x^w)^{M_w}$, which gives the following values of M_w when $w \leq 13$:

w	1	2	3	4	5	6	7	8	9	10	11	12	13
M_w	0	1	1	0	1	0	1	1	1	1	2	2	3

(8)

In the eight-loop computation that we present here, it is enough to find all relations up to transcendentality 13. This is easily done by brute force based on shuffle and stuffle relations. In `zetafunctions.m`, the result is saved into the substitution rules `subZetaReduce #`. We obtain that the following MZVs are independent:

$$\zeta_2 = \frac{\pi^2}{6}, \quad \zeta_w \text{ for } w \text{ odd}, \quad \zeta_{2,6}, \quad \zeta_{2,8}, \quad \zeta_{2,10}, \quad \zeta_{1,2,8}, \quad \zeta_{1,2,10}, \quad \zeta_{1,3,9}, \quad \zeta_{1,1,2,8}.$$
(9)

As expected, this is consistent with the values of M_w tabulated in (8). We will actually find out that the weak coupling perturbative expansion of the Konishi anomalous dimension (63) up to eight loops contains only a subset of the irreducible MZVs (9), namely it only involves $\zeta_{1,2,8}$ and single-indexed MZVs.

If one wants to go to higher weights, one can use more efficient algorithms based on remarkable relations derivable from the shuffle and stuffle algebras [69, 70, 71, 72]. Assuming diophantine conjecture, all the relations among MZVs were worked out in [72] up to weight 22 and, with some restrictions on the depth, to higher weights. These results are available online.

2.2. Multiple Hurwitz zeta functions (η -functions)

In the evaluation of Feynman integrals, one often encounters polylogarithms. These functions can be defined as a generalization of Euler-Zagier sums which preserve the shuffle but not the stuffle algebra, see e.g. [65].

⁴MZV is called irreducible if it cannot be written as a combination of MZVs of lower depth or weight.

⁵The marginally divergent case is not included in this statement, see Appendix A.1 for its treatment.

Quite remarkably, the perturbative weak coupling solution of the FiNLIE is expressed in terms of functions which are in a sense complementary to polylogarithms: they are generalization of Euler-Zagier sums which preserve the stuffle but not the shuffle algebra. They are defined as follows

$$\eta_{a_1, a_2, \dots, a_k}(u) = \sum_{0 \leq n_1 < n_2 < \dots < n_k < \infty} \frac{1}{(u + i n_1)^{a_1} (u + i n_2)^{a_2} \dots (u + i n_k)^{a_k}}. \quad (10)$$

We introduced $i = \sqrt{-1}$ and started the sum from 0 so that these functions have poles at position 0, $-i$, $-2i$, \dots . This way, u will coincide with the spectral parameter of the AdS/CFT integrable model. This sum is defined under the same restrictions on a_i as for the case of Euler-Zagier sums.

The MZVs are then related to the functions (10) evaluated at the point $u = i$:

$$\zeta_{a_1, a_2, \dots, a_k} = i^{-\sum_{j=1}^k a_j} \eta_{a_1, a_2, \dots, a_k}(i). \quad (11)$$

The single-indexed functions (10) differ from the Hurwitz zeta function $\zeta_a(u) = \sum_{n \geq 1} (u + a)^{-n}$ only by a slight change of variable, and they are related to derivatives of the polygamma function ψ by

$$\eta_a(u) = \frac{i^a}{(a-1)!} \psi^{(a-1)}(-i u). \quad (12)$$

Hence, the multiple-indexed functions (10) can be called multiple Hurwitz zeta functions or multiple polygamma functions. These sums can be also recast in terms of infinite cyclotomic harmonic sums defined e.g. in [73]. To simplify further presentation, we will often refer to (10) as η -functions.

In Appendix A.1, which defines marginally divergent η -functions, we choose to enforce (12) even when $a = 1$, which means that we define $\eta_1(u) \equiv i\psi(-i u)$. Relation (11) is also enforced for marginally divergent case, hence one defines $\zeta_1 \equiv \gamma_{\text{Euler-Mascheroni}}$. A number of the FiNLIE functions do depend on this regularization prescription, however it is a good check of our computation that the physical quantities, like the anomalous dimension, do not depend on the regularization.

In the following we will denote multi-indices by the capital letters I, J , and K . One additional useful property of η -functions, which holds in particular for regularized sums is

$$\eta_{a, I} - \eta_{a, I}^{[2]} = \frac{1}{u^a} \eta_I^{[2]}, \quad \text{where } \eta_I^{[n]} \equiv \eta_I(u + i n/2). \quad (13)$$

η -functions have an infinite ladder of poles at $u = -in$, $n \in \mathbb{Z}_+$. In the FiNLIE, this structure of poles is natural at weak coupling $g \ll 1$: in this limit, the Zhukovsky cuts on the intervals $[-2g - in, 2g - in]$ collapse into poles. In the functions entering the FiNLIE, one can in principle expect additional poles at shifted Bethe roots, i.e. at $u = u_j(g) - in$. For the Konishi state, there are two Bethe roots, $u_1(g) = -u_2(g)$, and $u_1(0) = \frac{1}{\sqrt{12}}$. At weak coupling, the appearance of such poles would indicate the presence of objects like $\eta_I(u + u_j(0))$ or even of more complicated functions of type (16). However, we experimentally observe a fine tuning which results in the cancellation of such ladders of poles at shifted Bethe roots, even though a finite number of poles may survive for some quantities. Therefore, the possible class of functions entering the FiNLIE solution (at weak coupling) is strongly constrained, in the sense that the poles of η -functions correspond only to the collapse of Zhukovsky cuts. Further discussion of this question is given in section 3.6.

2.3. Integrals involving η -functions

The integrals we encounter in this work reduce to a sum of the following elementary ones:

$$\int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \bar{\eta}_I^{[-2]} \frac{1}{(u-v)^a} \eta_J^{[2]}, \quad (14)$$

where $\bar{\eta}$ -functions are the complex conjugate of η -functions, in the sense that $\bar{\eta}_I(u) \equiv (\eta_I(u^*))^*$, where $*$ denotes the complex conjugation.

An important property of this integral is that it always evaluates in terms of η -functions and MZVs. For example, for $\text{Im}(v) > 0$:

$$\int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \bar{\eta}_3^{[-2]} \frac{1}{(u-v)} \eta_2^{[2]} = -\frac{6}{5} \frac{\zeta_2^2}{v} - i \frac{\zeta_3}{v^2} + 3i \zeta_3 \eta_2(v) + \zeta_2 \eta_3(v) + \eta_{3,2}(v), \quad (15)$$

and in the limit $v \rightarrow 0 + i0$ this integral evaluates to $-\frac{i}{2}\zeta_5$. #

To prove this property, we introduce an intermediate function that will be called generalized η -function⁶:

$$\eta_{a_1, a_2, \dots, a_k}^{\{v_1, v_2, \dots, v_k\}}(u) \equiv \sum_{0 \leq n_1 < n_2 < \dots < n_k < \infty} \prod_{j=1}^n \left(\frac{1}{u + i n_j - v_j} \right)^{a_j}. \quad (16)$$

⁶Generalized $\bar{\eta}$ -functions are defined by $\bar{\eta}_I^{\{v_1, \dots\}}(u) \equiv \left(\eta_I^{\{v_1^*, \dots\}}(u^*) \right)^*$.

The integral (14) is evaluated iteratively by taking residues: if the multi-index I has the form $I = b, \check{I}$, and if $0 < \text{Im}(v) < 1$, one has⁷

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \bar{\eta}_I^{[-2]} \frac{1}{(u-v)^a} \eta_J^{[2]} \\
&= \sum_{k=1}^{\infty} \int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \left(\frac{\bar{\eta}_{\check{I}}^{[-2]}}{u^b} \right)^{[-2k]} \frac{1}{(u-v)^a} \eta_J^{[2]} \\
&= \sum_{k=1}^{\infty} \int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \left(\frac{\eta_J^{[2]}}{(u-v)^a} \right)^{[+2k]} \frac{\bar{\eta}_{\check{I}}^{[-2]}}{u^b} - \text{Res}_{u=v} \left(\frac{\bar{\eta}_{\check{I}}^{[-2]}}{u^b} \right)^{[-2k]} \frac{\eta_J^{[2]}}{(u-v)^a} \\
&= \int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \frac{\left(\eta_{a,J}^{\{v,0,0,\dots\}} \right)^{[+2]} \bar{\eta}_{\check{I}}^{[-2]}}{(u+i0)^b} - \text{Res}_{u=v} \bar{\eta}_{\check{I}}^{[-2]} \frac{1}{(u-v)^a} \eta_J^{[2]}. \tag{17}
\end{aligned}$$

On the one hand, the residue in (17) is expressed in terms of η -functions at point $u = v$ (because the derivatives of η -functions are themselves η -functions). On the other hand, the integral term gives rise to two different cases: if $\check{I} = \emptyset$, then by closing the integration contour upwards this integral evaluates to $-\text{Res}_{u=0} \frac{\left(\eta_{a,J}^{\{v,0,0,\dots\}} \right)^{[+2]} \bar{\eta}_{\check{I}}^{[-2]}}{u^b}$, whereas if \check{I} has at least one element, it can be written as $\check{I} = c, \check{\check{I}}$, allowing to move the integration contour again to get

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \frac{\left(\eta_{a,J}^{\{v,0,0,\dots\}} \right)^{[+2]} \bar{\eta}_{\check{I}}^{[-2]}}{(u+i0)^b} \\
&= \int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \frac{\left(\eta_{b,a,J}^{\{0,v,0,0,\dots\}} \right)^{[+2]} \bar{\eta}_{\check{\check{I}}}^{[-2]}}{(u+i0)^c} - \text{Res}_{u=0} \frac{\left(\eta_{a,J}^{\{v,0,0,\dots\}} \right)^{[+2]} \bar{\eta}_{\check{I}}^{[-2]}}{u^b}. \tag{18}
\end{aligned}$$

If the initial multi-index I has n elements, the process stops after n iterations, and one gets the integral (14) in terms of standard η -functions evaluated either at point $u = i$ (where they are equal to MZVs (11)) or at point $u = v$, and of generalized η -functions of the form $\eta_K^{\{\dots,0,0,v,0,0,\dots\}}$, evaluated at point $u = i$.

⁷For simplicity, we assume that the integral is convergent. In practice, sometimes one needs to combine few elementary integrands to achieve convergence.

As explained in Appendix A.2, the generalized η -functions of type $\eta_K^{\{\dots,0,0,v,0,0,\dots\}}$ can be expressed through standard η -functions and $\bar{\eta}$ -functions evaluated at point v . Moreover, the result of integration is analytic above or below the contour of integration and hence cannot contain simultaneously η -functions and $\bar{\eta}$ -functions. For instance, in example (15) the answer is analytic in the upper half-plane and hence it should be expressible only through η -functions. Appendix A.3 gives an additional class of relations which allow one to express η -functions in terms of $\bar{\eta}$ -functions (or the opposite) and to insure this last property we discussed. This class of relations is also necessary to show that terms like the η -functions evaluated at Bethe roots cancel in the final expression for the anomalous dimension.

The integration algorithm and the properties of η -functions described above and in Appendix A are implemented in our *Mathematica* package [1] which was used to perform all analytical computations needed to solve the FiNLIE at weak coupling. Examples of usage are given in the file `usage.nb`.

3. Set of equations for weak coupling expansion

Our strategy for weak coupling expansion was summarized in [27]⁸. The goal is to determine three quantities ρ, ρ_2, U from which all the Y-system can be straightforwardly restored. As shown in figure 1 in [27], the set of equations divides into two iterative processes: on the one hand, there is an internal “Y-cycle” which has to run every time we want to compute one more order in g^2 for the densities ρ and ρ_2 . On the other hand, there is a “wrapping cycle”, which computes the function U , and which only needs to be run once for every four orders in g^2 .

Compared to the weak coupling expansion [27], which was performed up to six loops, we will make two essential updates. First, we add an equation (42) for an auxiliary quantity \hat{h} which becomes important at 7 loops. Second, we will compute the wrapping correction to U using (46) and (44), to reach the double-wrapping eight-loop order. By contrast, the asymptotic expression for U was sufficient at single-wrapping.

For the sake of completeness and to provide some technical details, we will write down all the necessary equations even though some of them were

⁸The reader might find it useful to take a look on [27] before going into details of this section

written in [54, 27]. The reader should however consult [27] or section 2 of [54] for the notations we use, in particular for the definition of integral kernels, Zhukovsky variables, mirror and magic sheets. The equations presented below can be used for a straightforward weak coupling expansion to an arbitrary order. Although the expansion is technically quite evolved, we developed mathematical techniques which allow handling it.

3.1. Restoring the Y- and T-systems from ρ , ρ_2 and U

The T-(Y-)systems involve infinitely many T-(Y-)functions, and an important point towards a finite set of equations is to rewrite these functions in terms of a finite number of Q-functions [57], which are themselves parameterized by a few densities [54]. In the finite set of equations (FinLIE) of [54], the T-functions are parameterized as follows by the functions ρ, ρ_2, U :

The T-functions of the right band ($\{T_{a,s} : s \geq a\}$) are computed in a certain gauge which is denoted by the letter \mathcal{T} . They are expressed in terms of the function ρ alone:

$$\mathcal{T}_{0,s} = 1, \quad \mathcal{T}_{1,s} = s + \mathcal{K}_s \hat{\star} \rho, \quad \mathcal{T}_{2,s} = \hat{\mathcal{T}}_{1,1}^{[+s]} \hat{\mathcal{T}}_{1,1}^{[-s]}. \quad (19)$$

On the other hand, the T-functions of the upper band ($\{T_{a,s} : a \geq |s|\}$) are computed in a different gauge denoted by the letter \mathcal{S} , and they are parameterized by a handful of q -functions including $U^2 = q_{123}/q_1$:

$$\begin{aligned} \mathcal{S}_{a,2} &= q_{\emptyset}^{[+a]} \bar{q}_{\emptyset}^{[-a]}, \\ \mathcal{S}_{a,1} &= q_1^{[+a]} \bar{q}_2^{[-a]} + q_2^{[+a]} \bar{q}_1^{[-a]} + q_3^{[+a]} \bar{q}_4^{[-a]} + q_4^{[+a]} \bar{q}_3^{[-a]}, \\ \mathcal{S}_{a,0} &= q_{12}^{[+a]} \bar{q}_{12}^{[-a]} + q_{34}^{[+a]} \bar{q}_{34}^{[-a]} - q_{14}^{[+a]} \bar{q}_{14}^{[-a]} - q_{23}^{[+a]} \bar{q}_{23}^{[-a]} - q_{13}^{[+a]} \bar{q}_{24}^{[-a]} - q_{24}^{[+a]} \bar{q}_{13}^{[-a]}, \\ \mathcal{S}_{a,-1} &= (U^{[+a]} \bar{U}^{[-a]})^2 \mathcal{S}_{a,1}, \\ \mathcal{S}_{a,-2} &= (U^{[a+1]} U^{[a-1]} \bar{U}^{[-a-1]} \bar{U}^{[-a+1]})^2 \mathcal{S}_{a,2}. \end{aligned} \quad (20)$$

These q -functions are expressed through the functions q_{12}, q_1, q_2 , and U as

follows

$$q_{13} = q_{12} \sum_{k=0}^{\infty} \left(\frac{U^2 q_1^2}{q_{12}^+ q_{12}^-} \right)^{[2k+1]}, \quad q_{23} = q_{14} = q_{12} \sum_{k=0}^{\infty} \left(\frac{U^2 q_1 q_2}{q_{12}^+ q_{12}^-} \right)^{[2k+1]}, \quad (21a)$$

$$q_{24} = q_{12} \sum_{k=0}^{\infty} \left(\frac{U^2 q_2^2}{q_{12}^+ q_{12}^-} \right)^{[2k+1]},$$

$$q_{34} = \frac{q_{13} q_{24} - q_{14} q_{23}}{q_{12}}, \quad q_{\emptyset} = \frac{q_2^- - q_2^+}{q_{12}}, \quad (21b)$$

$$q_3 = \frac{q_2 q_{13}^+ - q_{14}^+}{q_{12}^+}, \quad q_4 = \frac{q_2 q_{23}^+ - q_{24}^+}{q_{12}^+}.$$

The \mathcal{T} -gauge is chosen such that

$$q_1 = 1, \quad q_{12} = (u - u_1 - \alpha)(u + u_1 + \bar{\alpha}), \quad (22)$$

where u_1 is the exact Bethe root and α is a complex constant, which is adjusted to enforce the vanishing $\mathcal{T}_{1,0}(u_1 \pm i/2) = 0$. This constant has order $\alpha \propto g^8$ and is suppressed by one wrapping (asymptotically, we have $(q_{12})_{\text{as}} = Q \equiv u^2 - u_1^2$), hence it can be found iteratively⁹. Last, the function q_2 is parameterized by

$$q_2 = -iu + \mathcal{K} \hat{*} \rho_2 - \mathcal{K} * \hat{W}_{\text{pv}}, \quad (24)$$

where we define $W_a \equiv q_3^{[+a]} q_4^{[-a]} + q_4^{[+a]} q_3^{[-a]}$, $\hat{W} \equiv \lim_{a \rightarrow 0} W_a$ for $u \in \check{Z}$ (this function is then analytically continued to any complex value of u by avoiding short cuts) and $\hat{W}_{\text{pv}} \equiv \frac{1}{2}(\hat{W}^{[+0]} + \hat{W}^{[-0]})$ on the real axis. We will also use $W \equiv \lim_{a \rightarrow 0} W_a$ for $u \in \hat{Z}$ later on¹⁰. The term involving W is suppressed in wrapping and can be found recursively.

⁹More explicitly, $\mathcal{T}_{1,0}(u_1 + i/2) = 0$ implies the following equation:

$$\alpha = \frac{(\mathcal{T}_{1,0}(u_1 + \frac{i}{2}) - q_{1,2}(u_1 + i) \bar{q}_{1,2}(u_1)) + 2i u_1 \alpha^2 + \alpha \bar{\alpha} (-i + \alpha + 2u_1)(i + \bar{\alpha} + 2u_1)}{-4i(u_1 - \frac{i}{2})u_1}, \quad (23)$$

where the first term in the numerator is of order g^8 and the remaining ones are of order g^{16} .

¹⁰Note that $\hat{W}_{\text{pv}} = \hat{W} = q_3^{[+0]} q_4^{[-0]} + q_4^{[+0]} q_3^{[-0]}$ for $u \in \check{Z}$ but $\hat{W}_{\text{pv}} \neq W$ for $u \in \hat{Z}$.

The parameterization (24) is chosen in such a way that, on the one hand, the finiteness of the support of ρ_2 (i.e. the condition that $\rho_2 = 0$ on \hat{Z}_0) is equivalent to the condition $\mathcal{T}_{0,1}^c = 0$ (which is a consequence of \mathbb{Z}_4 -symmetry [54]), and that, on the other hand, ρ_2 is of the form $\sqrt{4g^2 - u^2}f(u)$, where f is an analytic function in the vicinity of the real axis¹¹. The latter property is a consequence of the built-in assumption that Q- and T-functions of the AdS/CFT integrable system have branch points of the second order.

The formulae presented above are nothing but an explicit realization of the Wronskian solution of the \mathbb{T} -hook [57] bisected into semi-infinite bands, without taking into consideration any gluing condition on these bands so far (i.e. that a global gauge exists in which the Hirota equation is satisfied everywhere on the \mathbb{T} -hook, including diagonals $a = \pm s$). The parameterization (19) and (24) was inspired by an approach in [74, 75]; it is just a suitable way to represent the Q-functions respecting basic analytical properties of the Y-system, with the details about particular state hidden into ρ and ρ_2 . Hence, so far we were exploiting merely algebraic properties of the Hirota equation, with minor analytical input. Now we come to more physical constraints which will determine U, ρ , and ρ_2 for the Konishi state.

3.2. *Y-cycle*

We use two integral equations for $Y_{1,1}$ and $Y_{2,2}$. First, an equation for the product $Y_{1,1}Y_{2,2}$ is obtained from the analyticity of the ratio $\frac{1}{Y_{1,1}Y_{2,2}} \frac{\mathcal{T}_{1,0}}{\mathcal{T}_{0,0}^-}$ in the upper half-plane which implies [54] the relation¹²

$$\frac{\log \left(\frac{1}{Y_{1,1}Y_{2,2}} \frac{\mathcal{T}_{1,0}}{\mathcal{T}_{0,0}^-} \right)}{\hat{x} - \frac{1}{\hat{x}}} = \mathcal{K} * 2\text{Re} \left(\frac{\log \left(\frac{\mathcal{T}_{1,0}}{\mathcal{T}_{0,0}^+} \right)}{\hat{x} - \frac{1}{\hat{x}}} \right)^{[-0]}. \quad (25a)$$

¹¹In [54], a slightly different parameterization was used in which ρ_2 was of the form $\sqrt{4g^2 - u^2}f(u) + (4g^2 - u^2)\tilde{f}(u)$. By contrast, the present convention incorporates the \tilde{f} -term into $\mathcal{K} * \hat{W}_{\text{pv}}$.

¹²The real part in the convolution was denoted as $\tilde{\eta}_b$ in [54].

Similarly, from the analyticity of the ratio $\frac{Y_{1,1}\mathcal{T}_{0,0}^-}{Y_{2,2}\mathcal{T}_{1,0}} \left(\frac{\mathcal{T}_{2,1}\mathcal{T}_{1,1}^-}{\mathcal{T}_{1,2}\mathcal{T}_{1,1}} \right)^2$ in the upper half-plane, one deduces that the following integral equation for $\frac{Y_{1,1}}{Y_{2,2}}$ holds:

$$\log \left(\frac{Y_{1,1}}{Y_{2,2}} \frac{\mathcal{T}_{2,1}^2}{\mathcal{T}_{1,2}^2} \right) = \log \left[\frac{\mathcal{T}_{1,0}}{\mathcal{T}_{0,0}^-} \left(\frac{\mathcal{T}_{1,1}^-}{\mathcal{T}_{1,1}} \right)^2 \right] - \mathcal{K} * 2i \operatorname{Im} \left(\log \left(\frac{\mathcal{T}_{1,0}}{\mathcal{T}_{0,0}^+} \right) - 2 \log \left(\frac{\mathcal{T}_{1,1}^+}{\mathcal{T}_{1,1}} \right) \right)^{[-0]}. \quad (25b)$$

As the asymptotic expressions

$$\begin{aligned} (Y_{1,1}Y_{2,2})_{\text{as}} &= \frac{B_{(-)}R_{(+)}}{B_{(+)}R_{(-)}}, \quad \left(\frac{Y_{1,1}}{Y_{2,2}} \frac{\mathcal{T}_{2,1}^2}{\mathcal{T}_{1,2}^2} \right)_{\text{as}} = \left(\frac{\Delta_{\text{as}}(u_1)}{\Delta_{\text{as}}(u_1) - 4} \right)^2 \frac{Q^+}{Q^-} \frac{B_{(+)}^{[-2]} x^{[-2]}}{B_{(-)}^{[+2]} x^{[+2]}}, \\ (\mathcal{T}_{a,0})_{\text{as}} &= Q^{[+a]} Q^{[-a]}, \quad (\mathcal{T}_{a,1})_{\text{as}} = a + \mathcal{K}_s \hat{*} (\rho_2)_{\text{as}}, \quad (\mathcal{T}_{1,s})_{\text{as}} = s + \mathcal{K}_s * (\rho)_{\text{as}}, \\ (\rho)_{\text{as}} &= 4 \frac{\sqrt{4g^2 - u^2}}{\Delta_{\text{as}}(u_1)}, \quad (\rho_2)_{\text{as}} = -4 \frac{\sqrt{4g^2 - u^2}}{\Delta_{\text{as}}(u_1) - 4}, \end{aligned} \quad (26)$$

with Δ_{as} defined in (60), solve (25a) and (25b), we subtract them from the exact quantities in (25) and get equations more suitable for weak coupling expansion.

Let us introduce the notation $(F)_{\text{r}} = \frac{F}{(F)_{\text{as}}}$ and define some standard combinations

$$H = \log \left(\frac{\mathcal{T}_{1,0}}{\mathcal{T}_{0,0}^+} \right)_{\text{r}}, \quad r = \log \left(\frac{\mathcal{T}_{1,1}}{\mathcal{T}_{1,1}} \right)_{\text{r}}, \quad \hat{r}_* = \log \left(\frac{\hat{q}_0}{\hat{\mathcal{T}}_{1,1}} \right)_{\text{r}} \quad (27)$$

which will be used throughout the text.

By subtracting the asymptotic quantities (26) in (25), one gets

$$\log (Y_{1,1}Y_{2,2})_{\text{r}} = \bar{H} - \left(\hat{x} - \frac{1}{\hat{x}} \right) \mathcal{K} * \left(\frac{\bar{H}^{[+0]}}{\hat{x}^{[+0]} - \frac{1}{\hat{x}^{[+0]}}} + \frac{H^{[-0]}}{\hat{x}^{[-0]} - \frac{1}{\hat{x}^{[-0]}}} \right), \quad (28a)$$

$$\log \left(\frac{Y_{1,1}}{Y_{2,2}} \frac{\mathcal{T}_{2,1}^2}{\mathcal{T}_{1,2}^2} \right)_{\text{r}} = \bar{H} + 2r^- + \mathcal{K} * \left(H^{[-0]} - \bar{H}^{[+0]} + 2r^{[1-0]} - 2r^{[-1+0]} \right). \quad (28b)$$

The equations (25) and (28) are valid in the upper half-plane. The latter ones are used in derivation of equation (46) on $(U)_r$. Together with their complex

conjugate, they also determine $(Y_{2,2}^+/Y_{2,2}^-)_r$ which appears in the exact Bethe equations (50).

To find the densities ρ and ρ_2 , one needs in particular to analytically continue (28) to the domain where u is real and belongs to $[-2g, 2g]$. In that case, the analytical continuation picks a residue which cancels some inhomogeneous terms in the equations and one gets¹³:

$$\log (Y_{1,1}Y_{2,2})_r = -4i g \sqrt{1-z^2} \int_{-\infty-i0}^{+\infty-i0} \frac{dv}{-2\pi i} \frac{1}{(2gz)^2 - v^2} \frac{H}{\sqrt{1 - \frac{4g^2}{v^2}}}, \quad (29a)$$

$$\log \left(\frac{Y_{1,1}}{Y_{2,2}} \frac{\mathcal{T}_{2,1}^2}{\mathcal{T}_{1,2}^2} \right)_r = \int_{-\infty-i0}^{+\infty-i0} \frac{dv}{-2\pi i} \frac{2v}{(2gz)^2 - v^2} (H + 2r^+) , \quad (29b)$$

where we use $z = u/2g$, and the weak coupling expansion is to be performed in the double scaling regime: $g, u \rightarrow 0$ with z being finite. In (29), H (resp. r^+) implicitly denotes $H(v)$ (resp. $r(v + i/2)$), and starting from now, for all functions that appear inside an integral, if the argument of a function is not specified then it is the integration variable.

Equations (25) on $Y_{1,1}, Y_{2,2}$ correspond to the physical constraints, and in particular one can show their equivalence to the corresponding TBA equations [54]. On the other hand, there is a purely algebraic constraint which tells us that $Y_{1,1}, Y_{2,2}$ can be expressed through ρ, ρ_2 , and U (where U is suppressed in wrapping and in this sense is known). Explicitly

$$\frac{1 + Y_{2,2}}{1 + Y_{1,1}^{-1}} = \frac{\mathcal{T}_{2,2}^+ \mathcal{T}_{2,2}^-}{\mathcal{T}_{3,2} \mathcal{T}_{1,2}} / \frac{\mathcal{T}_{1,1}^+ \mathcal{T}_{1,1}^-}{\mathcal{T}_{1,0} \mathcal{T}_{1,2}} = \frac{q_\emptyset^+ \bar{q}_\emptyset^- \mathcal{T}_{1,0}}{\mathcal{T}_{1,1}^+ \mathcal{T}_{1,1}^-}, \quad (30a)$$

$$\frac{1 + Y_{2,2}^{-1}}{1 + Y_{1,1}} = \frac{\mathcal{T}_{2,2}^+ \mathcal{T}_{2,2}^-}{\mathcal{T}_{2,1} \mathcal{T}_{2,3}} / \frac{\mathcal{T}_{1,1}^+ \mathcal{T}_{1,1}^-}{\mathcal{T}_{0,1} \mathcal{T}_{2,1}} = \frac{\hat{\mathcal{T}}_{1,1}^{[1+0]} \hat{\mathcal{T}}_{1,1}^{[-1-0]}}{\mathcal{T}_{1,1}^+ \mathcal{T}_{1,1}^-}. \quad (30b)$$

One can check that these equations are also satisfied by the asymptotic solution (26), so we also subtract the asymptotic quantities to prepare for a weak coupling expansion. Finally, by excluding $Y_{1,1}$ and $Y_{2,2}$ from (30) and

¹³We use the parity $\mathcal{T}_{a,s}(-u) = \mathcal{T}_{a,s}(u)$, $\mathcal{T}_{a,s}(-u) = \mathcal{T}_{a,s}(u)$, which is satisfied for the Konishi state, and the reality of T-functions in the chosen gauges. These properties imply that $H(-u) = \bar{H}(u)$ and $r(u) = \bar{r}(u) = r(-u)$.

(29) and performing some algebra one gets

$$\begin{aligned} \hat{r}_*^{[1+0]} + \hat{r}_*^{[-1-0]} - r^+ - r^- + \log(\mathcal{T}_{1,0})_r = \\ -4ig\sqrt{1-z^2} \int_{-\infty-i0}^{+\infty-i0} \frac{dv}{-2\pi i} \frac{1}{(2gz)^2 - v^2} \frac{H}{\sqrt{1 - \frac{4g^2}{v^2}}}, \end{aligned} \quad (31a)$$

$$\begin{aligned} \hat{r}_*^{[1+0]} + \hat{r}_*^{[-1-0]} + r^+ + r^- + \log(\mathcal{T}_{1,0})_r + 2 \log \left(\frac{\rho}{\rho_2 + W - \hat{W}_{\text{pv}}} \right)_r = \\ + \int_{-\infty-i0}^{+\infty-i0} \frac{dv}{-2\pi i} \frac{2v}{(2gz)^2 - v^2} (H + 2r^+) . \end{aligned} \quad (31b)$$

This is a set of two coupled equations which constitute the Y-cycle. They are used to find ρ and ρ_2 . To uniquely solve them, an additional constraint $\mathcal{T}_{1,1}(u_1) = 0$ should be imposed. An explicit expansion of these equations is worked out in the *Mathematica* notebook `WeakCoupling.nb` [1], while here we will discuss an instructive simplified example, by putting $W_a = U = \alpha = H = 0$. We have put to zero the quantities that are of order g^8 , i.e. the quantities which are responsible for the wrapping corrections. In this approximation we have $\hat{r}_* = \hat{r}$, and the equations (31) reduce to

$$\hat{r}^{[1+0]} + \hat{r}^{[-1-0]} - \hat{r}^{[1-0]} - \hat{r}^{[-1+0]} = 0, \quad (32a)$$

$$\begin{aligned} \hat{r}^{[1+0]} + \hat{r}^{[-1-0]} + \hat{r}^{[1-0]} + \hat{r}^{[-1+0]} \\ + 2 \log \left(\frac{\rho}{\rho_2} \right)_r - \int_{-\infty-i0}^{+\infty-i0} \frac{dv}{-2\pi i} \frac{4v \hat{r}^+}{(2gz)^2 - v^2} = 0, \end{aligned} \quad (32b)$$

with

$$\hat{r} = \log \left(\frac{\hat{q}_2^{[+1]} - \hat{q}_2^{[-1]}}{1 + \hat{c}^{[+1]} - \hat{c}^{[-1]}} \right)_r, \quad (33)$$

where $\hat{q}_2 = -iu + \mathcal{K}\hat{*}\rho_2$, $(\hat{q}_2)_{\text{as}} = -iu + i\frac{c_1}{g\hat{x}}$, $\hat{c} = \mathcal{K}\hat{*}\rho$ and $(\hat{c})_{\text{as}} = i\frac{g c_2}{\hat{x}}$ are defined everywhere in the u -plane outside the cut \hat{Z}_0 . The constants c_i are given by $c_1 = -\frac{4g^2}{\Delta_{\text{as}}-4}$, $c_2 = \frac{4}{\Delta_{\text{as}}}$, but we will only use that $c_i \propto \mathcal{O}(g^0)$.

Next, we define perturbations $\delta\hat{q}_2 = \hat{q}_2 - (\hat{q}_2)_{\text{as}}$, $\delta\hat{c} = \hat{c} - (\hat{c})_{\text{as}}$, $\delta\rho_2 = \rho_2 - (\rho_2)_{\text{as}}$, $\delta\rho = \rho - (\rho)_{\text{as}}$ and expand the equations to the linear order in

the perturbations and to the leading nontrivial order in g . To perform the expansion, one uses firstly that close to the cut one has

$$(\hat{q}_2)_{\text{as}}^{[\pm 0]} = i \frac{c_1}{g \hat{x}^{[\pm 0]}} + \mathcal{O}(g^1), \quad (\hat{c})_{\text{as}}^{[\pm 0]} = i \frac{g c_2}{\hat{x}^{[\pm 0]}} + \mathcal{O}(g^3), \quad (34)$$

with $\hat{x}^{[\pm 0]} = 1/\hat{x}^{[\mp 0]} = z \pm i\sqrt{1-z^2}$, and

$$\delta \hat{c}^{[\pm 0]}(z) = \pm \frac{1}{2} \delta \rho(z) + \oint_{-1}^1 \frac{d\tilde{z}}{-2\pi i} \frac{\delta \rho(\tilde{z})}{z - \tilde{z}}, \quad (35)$$

where we view $\delta \rho$ as a function of $z = u/2g \in [-1, 1]$. One will find that only a certain combination of $\delta \hat{c}$, namely $\delta \hat{c}^{[+0]} - \delta \hat{c}^{[-0]} = \delta \rho$, appears at the leading order. For $\delta \hat{q}_2^{[\pm 0]}$ one can use the representation which is analogous to (35), however it is more instructive to keep $\delta \hat{q}_2^{[\pm 0]}$ as it is.

Secondly, one has the following perturbative expansion if the argument of a function is far from the cut:

$$(\hat{q}_2)_{\text{as}}(u) = -i u + i \frac{c_1}{u} + \mathcal{O}(g^1), \quad (\hat{c})_{\text{as}} = \mathcal{O}(g^2),$$

$$\delta \hat{q}_2(u) = \frac{2g}{-2i} \int_{-1}^1 \frac{dw}{\pi} \frac{\delta \rho_2(w)}{u - 2gw} = \frac{i g}{u} M[\delta \rho_2] + \dots, \quad (36)$$

where we defined $M[f] \equiv \int_{-1}^1 \frac{dz}{\pi} f(z)$.

By comparing (35) to (36), we see that, far from the cut, these functions are suppressed by one order of g compared to the same functions close to the cut. In particular, $\delta \hat{c}$ with its argument far from the cut does not appear in the leading order expansion.

The argument of a function is far from the cut in two situations. The first one is when one encounters a function with a shift: $f^{[\pm 2]} = f(2gz \pm i) = f(\pm i) + \dots$. The second one is when we integrate over the argument, as in the integral in (32b). The contour of integration can be deformed such that it never approaches the branch points $u = \pm 2g$. Hence one can apply (36) and take the integral in (32b) explicitly.

By implementing the outlined strategy, one gets at the leading order

$$(\hat{x}^{[+0]})^2 \delta \hat{q}_2^{[+0]} - (\hat{x}^{[-0]})^2 \delta \hat{q}_2^{[-0]} = \lambda_1 \delta \rho = i \lambda_2 M[\delta \rho_2] (\hat{x}^{[+0]} - \hat{x}^{[-0]}), \quad (37)$$

where $\lambda_1 = \frac{c_1(2c_1 c_2 - 4c_1 - 1)}{c_2(1 - 2c_1)}$ and $\lambda_2 = \frac{c_1(2c_1 c_2 - 4c_1 - 1)}{(1 + 4c_1)(c_1 c_2 - c_1 - 1)}$.

Note that the equation $(\hat{x}^{[+0]})^2 \delta \hat{q}_2^{[+0]} - (\hat{x}^{[-0]})^2 \delta \hat{q}_2^{[-0]} = 0$ always has a solution $\delta \hat{q}_2 = \frac{A}{\hat{x}^2}(\hat{x} + \frac{1}{\hat{x}})$ which has good analytic and parity properties (it decreases at infinity and it is antisymmetric on the magic sheet). Hence, for any constant \tilde{M} one can solve (37) by

$$\delta \hat{q}_2 = \frac{i \lambda_2 \tilde{M}}{\hat{x}} + \frac{A}{\hat{x}^2} \left(\hat{x} + \frac{1}{\hat{x}} \right), \quad (38)$$

where A is adjusted such that $M[\delta \rho_2] = \tilde{M}$. This one-parametric ambiguity is fixed by recalling that one should impose $\mathcal{T}_{1,1}(u_1) = 0$. Indeed, since $(\mathcal{T}_{1,1})_{\text{as}}$ vanishes at Bethe roots by construction, one should impose that the correction to $(\mathcal{T}_{1,1})_{\text{as}}$ obeys:

$$\delta \mathcal{T}_{1,1}(u_1) = \delta q_2^+ + \delta \bar{q}_2^- + W_1|_{u=u_1} = \frac{g}{u_1^2 + \frac{1}{4}} M[\delta \rho_2] + \dots = 0, \quad (39)$$

where \dots denotes corrections of higher order in g . These corrections define the value of $M[\delta \rho_2]$, and in particular the magnitude of $\delta \rho_2$. In our simplified case $W_1 = 0$ and hence $M[\delta \rho_2] = 0$, which means, due to (37), that $\delta \rho_2 = \delta \rho = 0$. This is only expected because the asymptotic solution should be exact when we set to zero all the terms that are responsible for wrapping. Although the solution we've got is trivial, its derivation allowed us to demonstrate the essential features of the perturbative expansion of (31).

When we add the wrapping corrections, they produce non-zero terms in the r.h.s. of (32) which are suppressed in powers of g and induce corrections to the asymptotic expressions for ρ and ρ_2 . At the leading order one has

$$\begin{aligned} \delta \rho &= g^9 \sqrt{1 - z^2} (-162 - 432 \zeta_3 + 432 \zeta_5 + 504 \zeta_7), \\ \delta \rho_2 &= g^7 \sqrt{1 - z^2} \left[-378 - 864 \zeta_3 + \frac{72}{5} \zeta_2^2 + 48 \zeta_3^2 + \frac{656}{35} \zeta_2^3 + 624 \zeta_5 + 168 \zeta_7 \right. \\ &\quad \left. + \text{Im}(-144\sqrt{3}\eta_2^+ + 144\eta_3^+ - 24\sqrt{3}\eta_4^+) + z^2 \left(432 - 432 \zeta_2 \right. \right. \\ &\quad \left. \left. + 144 \zeta_3 + \frac{864}{5} \zeta_2^2 + 576 \zeta_3^2 + \frac{7872}{35} \zeta_2^3 - 1536 \zeta_5 - 672 \zeta_7 \right) \right], \end{aligned} \quad (40)$$

where $\eta_a^+ \equiv \eta_a(\frac{1}{\sqrt{12}} + \frac{i}{2})$.

Since $\frac{\rho(u)}{\sqrt{4g^2 - u^2}}$ and $\frac{\rho_2(u)}{\sqrt{4g^2 - u^2}}$ are even functions analytic in the vicinity of the real axis, they are represented in the double scaling regime as finite degree

polynomials in z^2 at any given order of g , c.f. (40). This property eventually follows from (37) supplemented with sources from wrapping terms, and it can be used to transform (37) to algebraic equations for the coefficients of the polynomials.

In `WeakCoupling.nb` [1], we compute the first three orders of $\delta\rho$ and $\delta\rho_2$, $\delta\rho$ up to the order g^{13} and $\delta\rho_2$ up to the order g^{11} . These orders are necessary for our computation of the energy up to eight loops. One formally needs also the fourth order of $\delta\rho_2$, however it enters only as $M[\delta\rho_2]$, because of expansion similar to (36), and hence it is found from (39).

3.3. Wrapping cycle

An equation on the function U , which constitutes the wrapping cycle, is [54]:

$$\left[\frac{U}{\hat{h}} \frac{\hat{h}^{[2]}}{U^{[2]}} \right]^2 = \left(\frac{Y_{1,1} \mathcal{T}_{0,0}^-}{Y_{2,2} \mathcal{T}_{1,0}} \left[\frac{\mathcal{T}_{2,1} \mathcal{T}_{1,1}^-}{\mathcal{T}_{1,2} \mathcal{T}_{1,1}^-} \right]^2 \right) \times \left(\frac{Y_{1,1} Y_{2,2} \mathcal{T}_{0,0}^-}{\mathcal{T}_{1,0}} \right)^{[2]}, \quad (41)$$

where on the r.h.s we outlined two factors encountered previously in the text. Both of them are analytic in the upper half-plane, where (41) is defined.

This equation on U contains an auxiliary function \hat{h} which also appears in the Bethe equations and which is found from:

$$\log \hat{h} = -2 \log \hat{x} + \mathcal{Z}^* \log \left(\frac{\mathcal{F}^+(Y_{1,1} Y_{2,2} - 1)}{\rho} \right), \quad (42)$$

$$\mathcal{F}^+ = \Lambda_F \prod_j (i \cosh [\pi (u - u_j)]) \exp \left[\int_{\mathbb{R} \setminus [-2g, 2g]} \frac{dv}{2i} (\coth[\pi(u - v)] + \text{sign}[v]) \log [Y_{1,1} Y_{2,2}] \right]. \quad (43)$$

Equation (41) actually allows finding U only up to a normalisation which is fixed by the following additional condition:

$$U \bar{U} = \sqrt{\mathcal{T}_{0,0}^+ \mathcal{T}_{0,0}^-} \frac{1 - Y_{1,1} Y_{2,2}}{\rho_2 + W - \hat{W}_{\text{pv}}}, \quad u \in \hat{Z}. \quad (44)$$

For this reason the normalization constant Λ_F is inessential.

The asymptotic expressions for \hat{h} and U are given by

$$(\hat{h})_{\text{as}} = \hat{x}^{-2} \hat{\sigma}_1, \quad (U)_{\text{as}} = \Lambda_U \frac{B_{(-)}}{x} \left(\frac{B_{(-)}}{B_{(+)}} \right)^{\frac{D^2}{1-D^2}} \sigma_1, \quad (45)$$

where $\frac{\hat{\sigma}_1(u+i/2)}{\hat{\sigma}_1(u-i/2)} = \prod_{j=1}^2 \sigma_{\text{BES}}(u, u_j)$ is the BES dressing phase [5].

Let us sketch how these asymptotic expressions are derived. To this end, let us note that equation (42) is equivalent to the Riemann-Hilbert problem $\hat{h}^{[+0]} \hat{h}^{[-0]} = \frac{\mathcal{F}^+(1-Y_{1,1}Y_{2,2})}{\rho}$, under the boundary condition $\hat{h} \sim u^{-1-L/2}$ at $u \rightarrow \infty$. By considering \hat{h} as function of \hat{x} (which is justified because \hat{h} has only one cut on the physical sheet), and using the periodicity of \mathcal{F} , one can derive from this Riemann-Hilbert problem the equation $\frac{\hat{h}(\hat{x}^+) \hat{h}(\frac{1}{\hat{x}^+})}{\hat{h}(\hat{x}^-) \hat{h}(\frac{1}{\hat{x}^-})} = \frac{\rho^-}{\rho^+} \frac{1-Y_{1,1}^+ Y_{2,2}^+}{1-Y_{1,1}^- Y_{2,2}^-}$. For the asymptotic values (26) of ρ and Y -s, this is precisely the crossing equation [76] on the dressing phase analytically continued using the trick of [77]. This explains how the expression (45) for $(\hat{h})_{\text{as}}$ is obtained. To simplify (41) for asymptotic quantities and to find $(U)_{\text{as}}$, one uses the following remarkable relation: $\left(\frac{\mathcal{T}_{1,1}}{\overline{\mathcal{T}}_{1,1}} \right)_{\text{as}} = \hat{x}_1^+ \hat{x}_1^- \frac{B_{(+)}^- x^-}{B_{(-)}^+ x^+}$.

Like for the Y-cycle, we subtract the asymptotic solution from (41) to get

$$\log \left(\frac{\hat{h}^{[2]} U}{\hat{h} U^{[2]}} \right)_r = \int_{-\infty-i0}^{+\infty-i0} \left[\left(\frac{1}{u-v} - \frac{1}{v+u} \right) \left(\frac{H}{2} + r^+ \right) + \left(\frac{1}{u-v+i} - \frac{1}{u+v+i} \right) \frac{\left(-\frac{u+i}{g} \sqrt{1 - \frac{4g^2}{(u+i)^2}} \right) H}{\frac{v}{g} \sqrt{1 - \frac{4g^2}{v^2}}} \frac{H}{2} \right] \frac{dv}{-2i\pi}. \quad (46)$$

$$\log(\hat{h})_r = \mathcal{Z}^* \left[\log \left(\frac{Y_{1,1} Y_{2,2} - 1}{\rho} \right)_r - \frac{1}{2} \log(Y_{1,1} Y_{2,2})_r + \int \frac{dv}{2i} ((\coth(\pi(2gw-v)) - \coth(\pi v)) \log(Y_{1,1} Y_{2,2})_r)_{\text{pv}} \right], \quad (47)$$

where the expression $\log(Y_{1,1} Y_{2,2})_r$ should be computed in the double scaling

regime $u = 2gz$, using (29a). Note that

$$\log(Y_{1,1}Y_{2,2} - 1)_r = \left(\frac{Y_{1,1}Y_{2,2}}{Y_{1,1}Y_{2,2} - 1} \right)_{\text{as}} \log(Y_{1,1}Y_{2,2})_r + \dots, \quad (48)$$

and the subleading terms of this expansion (denoted by \dots) are not necessary for the eight-loop computation in this paper.

Finally, the equation for the normalization of U (44) should be transformed into an equation for the normalization of $(U)_r$, which requires to continue (46) to real values of u and to re-expand it in the double scaling regime $u = 2gz$. For the leading wrapping correction of U , this gives the following equation for the normalization of U :

$$\begin{aligned} \log(U^{[2]}\bar{U}^{[-2]})_r = & \log \left(\frac{1 - Y_{1,1}Y_{2,2}}{\rho_2 + W - \hat{W}_{\text{pv}}} \mathcal{T}_{1,0} \right)_r + r^+ + r^- \\ & + 2 \int_{-\infty-i0}^{+\infty-i0} \frac{dv}{-2\pi i} \left(\frac{H + 2r^+}{v} - \frac{iH}{1+v^2} \right). \end{aligned} \quad (49)$$

The dependence on z should be the same in the r.h.s and the l.h.s., which is used as a nontrivial check for our computations.

Let us note that $\log(\hat{h})_r = \text{constant} \cdot g^8 + \mathcal{O}(g^{10})$. Since an overall normalization of \hat{h} is irrelevant, $\log(\hat{h})_r$ does not contribute to the computation of U at eight loops. However, the terms in $\log(\hat{h})_r$ of order g^{10} and g^{12} are needed to find corrections to the Bethe equations, as we will see below.

3.4. Bethe equations

The Bethe equations can be written as

$$e^{\phi(u_j)} = 1, \quad \text{where } e^{\phi(u)} \equiv - \left(\frac{\hat{h}^-}{\hat{h}^+} \right)^2 \frac{Y_{2,2}^+ \mathcal{T}_{1,2}^+ \hat{\mathcal{T}}_{1,1}^{[-2]}}{Y_{2,2}^- \mathcal{T}_{1,2}^- \hat{\mathcal{T}}_{1,1}^{[+2]}}. \quad (50)$$

The asymptotic function $(\phi)_{\text{as}}$ precisely coincides with the logarithm of the Beisert-Staudacher asymptotic Bethe equations [4, 5]. The function ϕ is subject to corrections to its asymptotic value; these corrections are computed

from the following equation:

$$\delta\phi = \log \left(\left(\frac{\hat{h}^-}{\hat{h}^+} \right)^2 \frac{\sqrt{\frac{Y_{1,1}^+ Y_{2,2}^+}{Y_{1,1}^- Y_{2,2}^-}}}{\sqrt{\frac{\left(\frac{Y_{1,1}}{Y_{2,2}} \frac{\mathcal{T}_{2,1}^2}{\mathcal{T}_{1,2}^2}\right)^+}{\left(\frac{Y_{1,1}}{Y_{2,2}} \frac{\mathcal{T}_{2,1}^2}{\mathcal{T}_{1,2}^2}\right)^-}}} \frac{\mathcal{T}_{2,1}^+ \hat{\mathcal{T}}_{1,1}^{[-2]}}{\mathcal{T}_{2,1}^- \hat{\mathcal{T}}_{1,1}^{[+2]}} \right)_r. \quad (51)$$

The square root in the numerator (resp the denominator) is expressed from (29a) (resp from (29b)), and \hat{h} is expressed from (47), while $\hat{\mathcal{T}}_{1,1}$ and $\mathcal{T}_{2,1} = q_2^{[2]} + \bar{q}_2^{[-2]} + W_2$ are found from the Wronskian parameterization discussed in section 3.1.

What one really needs is the value of $\delta\phi$ at $u = u_1$ which we parameterize as follows:

$$\delta\phi(u_1) = g^8 m_1 + g^{10} m_2 + g^{12} m_3 + g^{14} m_4 + \dots \quad (52)$$

We explicitly computed m_1, \dots, m_4 which are used for our computation of the energy up to eight loops. Their expressions can be found in `WeakCoupling.nb` [1]. Using this data, one finds the exact position of the Bethe root if one notices that, since $\phi(u_1) = 0$, one has $\delta\phi(u_1) = \phi(u_1) - (\phi)_{\text{as}}(u_1) = -(\phi)_{\text{as}}(\tilde{u}_1 + \delta u_1)$. The latter equation is solved perturbatively as follows

$$u_1 = \tilde{u}_1 + i \left(-\frac{1}{9} g^8 - \frac{4}{9} g^{10} + 2g^{12} - 8g^{14} \right) (m_1 + g^2 m_2 + g^4 m_3 + g^6 m_4) + \mathcal{O}(g^{16}), \quad (53)$$

where \tilde{u}_1 is the solution of the asymptotic Bethe equation $(\phi)_{\text{as}}(\tilde{u}_1) = 0$. Its explicit expression is

$$\begin{aligned} \tilde{u}_1 = \frac{1}{\sqrt{3}} \left[\frac{1}{2} + 4g^2 - 10g^4 + 8g^6(7 + 3\zeta_3) + g^8(-461 - 240\zeta_5 - 144\zeta_3) \right. \\ + 4g^{10}(1133 + 252\zeta_3 + 378\zeta_5 + 630\zeta_7) \\ - 6g^{12}(7945 + 1556\zeta_3 + 48\zeta_3^2 + 1944\zeta_5 + 2772\zeta_7 + 4704\zeta_9) \\ + 24g^{14}(21577 + 4572\zeta_3 + 240\zeta_3\zeta_5 + 24\zeta_3^2 + 4784\zeta_5 \\ \left. + 5706\zeta_7 + 8064\zeta_9 + 13860\zeta_{11}) + \dots \right]. \quad (54) \end{aligned}$$

3.5. Summary of the perturbative expansion

To conclude, the analytic perturbative solution of the FiNLIE relies on a perturbative expansion of the Bethe root u_1 , the two densities ρ and ρ_2 , and the function U .

In practice, one computes the deviations from the asymptotic expressions: $\delta\rho = \rho - (\rho)_{\text{as}}$, $\delta\rho_2 = \rho_2 - (\rho_2)_{\text{as}}$, using equations (31), and $\log(U)_r = \log U / (U)_{\text{as}}$, using equations (46) and (44). These equations contain a number of auxiliary objects. Firstly, H, r, \hat{r}_* (defined in (27)), and $\log(\mathcal{T}_{1,0})_r$, W_a are computed from the Wronskian parameterization given in section 3.1. Secondly, \hat{h} and $\log(Y_{1,1}Y_{2,2})_r$ are found respectively from equations (47) and (29a). These auxiliary objects appear in the source terms of (31), (46), (44)¹⁴, and they turn out to be suppressed by one order of wrapping, hence their contribution to the solution of the FiNLIE at given order is obtained simply by knowing the solution of the FiNLIE at lower orders.

The position of the exact Bethe root constraints the solution through the conditions $\mathcal{T}_{1,0}^+(u_1) = 0$ and $\mathcal{T}_{1,1}(u_1) = 0$. Let us stress that the “asymptotic solutions” (26),(45) are defined using the exact Bethe root u_1 , as opposed to the solution \tilde{u}_1 of the asymptotic Beisert-Staudacher Bethe equation. This exact Bethe root is found from equation (50). The correction $u_1 - \tilde{u}_1$ to the position of the Bethe root is suppressed by wrapping and, again, it can be computed iteratively.

3.6. Analytical structure of functions

It might be not immediately clear that the equations presented above can be solved analytically at each order of the weak coupling expansion. We are going to show that this is indeed so and we will precise the class of functions which appear during this expansion.

Let us start with the integration of densities ρ and ρ_2 , with either a Cauchy kernel \mathcal{K} , like in (19) and (24), or a Zhukovsky kernel like in (42). As we saw, these densities are $\sqrt{1 - z^2}$ times functions which are, at each order in perturbation theory, polynomials in z . Actually, in the double scaling regime (when $z = u/2g$ is fixed while $u \ll 1$), which one has to use for integrands if they are integrated on the finite support $[-2g, 2g]$, any quantity has an expansion which is either a polynomial or a polynomial times square root (or

¹⁴The perturbative expansion of (44), after subtracting the asymptotic solution is expressible through these auxiliary quantities, c.f. eight-loop approximations (49) and (48).

the sum of these two cases). Such expressions can be explicitly convoluted against both the Cauchy and the Zhukovsky kernel. The result of integration is either a polynomial in z , if the value of integral is computed in the double scaling regime, or a rational function in u , if it is to be computed in the ordinary regime, i.e with $g \rightarrow 0$ keeping u fixed.

Let us now consider the weak coupling expansion of the quantities in the ordinary regime. It is easy to check that when expanding asymptotic quantities in this regime, one always gets expressions which can be represented as linear combinations of terms of the type $\bar{\eta}_I^{[-2]} \frac{1}{(u-v)^a} \eta_J^{[2]}$. In the following such linear combination will be called a standard-type expression. We will now discuss two non-trivial operations one encounters during solution of the FiNLIE at weak coupling and show that both of them keep us in the class of standard-type expressions¹⁵. Hence, in this way we will iteratively demonstrate that exact quantities are explicitly computable and that they are always expressible in terms of standard-type expressions.

The first operation is integration. Except for the integrals over the finite support $[-2g, 2g]$, which we discussed already, all the integrals in the weak coupling version of the FiNLIE have an integration contour from $-\infty$ to $+\infty$, parallel to the real axis. In section 2.3 and in Appendix A we showed how to integrate standard-type expression along such contour, and the result is always a standard type expression. It is easy to iteratively verify, starting from the asymptotic solution, that the integrands are always of the standard type, and probably the only troublesome place is the equation on \hat{h} , and more precisely the integration with $\coth(\pi(2gw - v))$. For this, we can note that

$$\coth(\pi(2gw - v)) = -(\eta_1^{[2]}(v - 2gw) + \bar{\eta}_1(u - 2gw)). \quad (55)$$

Hence, this \coth is expressed as a linear combination of η -functions at each order of expansion in g^2 . Since η -functions obey the stuffle algebra and hence form a ring, we see that at a given order in g^2 , multiplying (55) by a standard-type expression always gives a standard-type expression.

The second operation is a semi-infinite summation. One such summation is needed to compute U : if $f = \log U/U^{[2]}$ is known from (46) then $\log U = \sum_{k=0}^{\infty} f^{[2k]}$. The other semi-infinite sums are used to compute q_{13}, q_{14}, q_{24} according to (21a). In both of these cases, the sums are of the

¹⁵We do not discuss less complicated operations like algebraic manipulations, under which this class of functions is clearly stable.

type $\sum_{k=0}^{\infty} f^{[2k]}$ for some function f . One can already note that, in these sums, the corresponding functions f are analytic in the upper half-plane, hence the sums can involve η -functions, but they should be free of any $\bar{\eta}$. Now, if f -s contain only poles at positions $u = -i k$, then the sum reduces to a standard-type expression, e.g

$$\sum_{k=0}^{\infty} \frac{1}{(u + i k)^a} \eta_I^{[2k+2]} = \eta_{a,I}. \quad (56)$$

However, if there is a pole in another position, a more generic class of functions (16) may appear. In principle, one could have poles at Bethe roots, but we observe a remarkable cancellation of these poles for the physical solutions of the Y-system.

To give an example of how the cancellation comes out, let us study the computation of q_{13} at the leading order, i.e. for $U^2 = -\frac{2g^4}{u^2}$ and $q_{12} = Q = u^2 - u_1^2$, where $u_1 = -u_2 = \frac{1}{2\sqrt{3}}$.

$$\begin{aligned} q_{13} &= -2g^4 Q \sum_{k=0}^{\infty} \left(\frac{1}{u^2 Q^+ Q^-} \right)^{[2k+1]} \\ &= -18g^4 Q \sum_{k=0}^{\infty} \left(\frac{1}{u^2} + \frac{i}{2} \sum_{j=1}^2 \left(\frac{1}{u - \frac{i}{2} - u_j} - \frac{1}{u + \frac{i}{2} + u_j} \right) \right)^{[2k+1]} \\ &= -18g^4 (i u + Q \eta_2^+). \end{aligned} \quad (57)$$

We see that each pole at a position $u = u_j + i k$ appears in two successive terms of the sum which cancel each other. We checked this cancellation mechanism at the first five orders of the perturbative expansion (which was needed for computing the eight-loop anomalous dimension). Hence we conclude that at least at these orders the following equation, which is behind the cancellation mechanism, holds:

$$\left(\frac{\hat{U}^+}{\hat{U}^-} \right)^2 = - \frac{q_{12}^{[+2]}}{q_{12}^{[-2]}} \quad (58)$$

at the zeroes of q_{12} which are $u = u_1 + \alpha$ and $u = -u_1 - \bar{\alpha}$. At the first four orders, we have $q_{12} = Q$ and (58) is nothing but the asymptotic Bethe equation. At least at the fifth order, (58) is still true, though it is no longer

the asymptotic Bethe equation. It is not the exact Bethe equation either, although it is equivalent to it because it does not follow from the constraints $\mathcal{T}_{1,0}^+(u_1) = \mathcal{T}_{1,1}(u_1) = 0$ and it is satisfied only if u_1 is the exact Bethe root.

In order to have the same cancellation of poles in the sums expressing q_{14} and q_{24} , one should also require that

$$\hat{q}_2^+ = \hat{q}_2^- \quad \text{at points } u \in \{u_1 + \alpha, -u_1 - \bar{\alpha}\}. \quad (59)$$

In view of (21b), this requirement can be considered as a regularity condition on q_\emptyset . At first four orders it is equivalent to the equation $\mathcal{T}_{1,1}(u_1) = 0$, and we also verified perturbatively that (59) holds at the fifth order.

Under the assumption that (58) and (59) hold at any order of the perturbation theory, all semi-infinite sums for q_{ij} result in standard-type expressions only.

By analyzing how \hat{H} and \hat{r} appears in the r.h.s. of equation (46) for U , we see that the cancellation of poles in q_{ij} propagates to the statement that $\log U/U^{[2]}$ only has poles at position $u = -ik$, hence U is given by an expression of standard type.

Let us stress that we verified (58) and (59) at the first five orders, and then predicted the general form of the functions under the assumption that they hold at any order. These two equations look very natural, since they ensure a more regular structure of q -functions, by cancelling out some ladders of poles. If they are not satisfied, then arbitrary generalized η -functions (16) would appear. This class of functions still forms a ring which insures that at most generalized η -functions are present in the answer at any order; however the answer would be considerably more complicated.

This cancellation of poles at shifted Bethe roots does not mean that η -functions *evaluated* at Bethe roots never appear. Such numerical constants appear in various places, e.g. in the constraints $\mathcal{T}_{1,0}^+(u_1) = 0$ and $\mathcal{T}_{1,1}(u_1) = 0$. For instance, they are already present in the leading displacement of the Bethe root. The statement that we discussed above is merely that there are no η -functions that depend *simultaneously* on the Bethe root and on the spectral parameter.

Since the computation of energy using (60) reduces to the computation of integrals of standard-type expressions, we conclude that if the constraints (58) and (59) hold, then the perturbative expansion of the energy is always given only by MZV-s and by η -functions evaluated at Bethe roots. However, η -functions at Bethe roots are expected to cancel [15]. We observe that they cancel indeed, at least up to eight loops.

4. Anomalous dimension

The anomalous dimension $\gamma_{\text{Konishi}} = \Delta_{\text{Konishi}} - 4$ is found from $\Delta_{\text{Konishi}} = \Delta_{\text{as}}(u_1) + \Delta_{\text{wrap}}$, where

$$\Delta_{\text{as}}(u_1) = 4 - 8g \operatorname{Im} \left(\frac{1}{\hat{x}_1^+} \right), \quad \Delta_{\text{wrap}} = \int_{\mathbb{R}-i0} \frac{-H(u)du}{\pi \sqrt{1 - \frac{4g^2}{u^2}}}. \quad (60)$$

As compared to the prediction from the asymptotic Bethe Ansatz, Δ_{as} receives the following corrections through the displacement of the Bethe roots:

$$\Delta_{\text{as}}(u_1) = \Delta_{\text{as}}(\tilde{u}_1) + \frac{4i}{\sqrt{3}} g^{10} (1 - 5g^2 + 14g^4 - 461g^6) \sum_{k=1}^4 m_k g^{2k-2} + \mathcal{O}(g^{18}). \quad (61)$$

The perturbative solution of the asymptotic Bethe equation gives

$$\begin{aligned} \Delta_{\text{as}}(\tilde{u}_1) = & 4 + 12g^2 - 48g^4 + 336g^6 - 12g^8(235 + 24\zeta_3) \\ & + 12g^{10}(2209 + 360\zeta_3 + 240\zeta_5) \\ & - 12g^{12}(22429 + 4608\zeta_3 + 3672\zeta_5 + 2520\zeta_7) \\ & + 24g^{14}(119885 + 29064\zeta_3 + 144\zeta_3^2 + 24156\zeta_5 \\ & \quad + 19656\zeta_7 + 14112\zeta_9) \\ & - 12g^{16}(2654761 + 742680\zeta_3 + 5760\zeta_3\zeta_5 + 6624\zeta_3^2 + 623904\zeta_5 + \\ & \quad 528552\zeta_7 + 447552\zeta_9 + 332640\zeta_{11}). \end{aligned} \quad (62)$$

Separately $\Delta_{\text{as}}(u_1)$ and Δ_{wrap} depend on η -functions evaluated at Bethe roots, however this dependence cancels out in their sum and one gets the

result which only involves Euler-Zagier sums: #

$$\begin{aligned}
\Delta_{\text{Konishi}} = & 4 + 12 g^2 - 48 g^4 + 336 g^6 + 96 g^8 (-26 + 6 \zeta_3 - 15 \zeta_5) \\
& - 96 g^{10} (-158 - 72 \zeta_3 + 54 \zeta_3^2 + 90 \zeta_5 - 315 \zeta_7) \\
& - 48 g^{12} (160 + 5472 \zeta_3 - 3240 \zeta_3 \zeta_5 + 432 \zeta_3^2 \\
& \quad - 2340 \zeta_5 - 1575 \zeta_7 + 10206 \zeta_9) \\
& + 48 g^{14} (-44480 + 108960 \zeta_3 + 8568 \zeta_3 \zeta_5 - 40320 \zeta_3 \zeta_7 - 8784 \zeta_3^2 \\
& \quad + 2592 \zeta_3^3 - 4776 \zeta_5 - 20700 \zeta_5^2 - 26145 \zeta_7 - 17406 \zeta_9 + 152460 \zeta_{11}) \\
& + 48 g^{16} (1133504 + 263736 \zeta_2 \zeta_9 - 1739520 \zeta_3 - 90720 \zeta_3 \zeta_5 \\
& \quad - 129780 \zeta_3 \zeta_7 + 78408 \zeta_3 \zeta_8 + 483840 \zeta_3 \zeta_9 + 165312 \zeta_3^2 \\
& \quad - 82080 \zeta_3^2 \zeta_5 + 41472 \zeta_3^3 + 178200 \zeta_4 \zeta_7 - 409968 \zeta_5 + 121176 \zeta_5 \zeta_6 \\
& \quad + 463680 \zeta_5 \zeta_7 + 49680 \zeta_5^2 + 455598 \zeta_7 + 194328 \zeta_9 - 555291 \zeta_{11} \\
& \quad - 2208492 \zeta_{13} - 14256 \zeta_{1,2,8}) \\
& + \mathcal{O}(g^{18}).
\end{aligned} \tag{63}$$

We will now discuss some checks of the consistency of this result. In the next paragraphs we will discuss the order of magnitude of the answer, and then show to what extent this answer can be predicted from the existing numeric data that gives the anomalous dimension for various values of the coupling. Another check will be given in the next section, where the terms having the highest transcendentality are derived.

The coefficients of the expansion (63) evaluate numerically to

$$\begin{aligned}
\Delta_{\text{Konishi}} \simeq & 4 + 0.75000 (4g)^2 - 0.18750 (4g)^4 + 0.08203 (4g)^6 - 0.05030 (4g)^8 \\
& + 0.03578 (4g)^{10} - 0.02728 (4g)^{12} + 0.02175 (4g)^{14} - 0.01791 (4g)^{16}.
\end{aligned} \tag{64}$$

It is expected (see e.g. section 3 of [44]) that the radius of convergency of weak coupling expansions in the AdS/CFT integrable system is at most $1/4$, because various Zhukovsky branch points in the u -plane collide when $g = \pm i/4$ ¹⁶. The coefficients in (64) are in agreement with this expectation.

One can further verify our results numerically using the 5-digit precision data for the anomalous dimension [78] which is given in figure 1. A Pad

¹⁶There might be other types of singularities which depend on quantum numbers. For instance, the radius of convergency shrinks to zero when the spin quantum number approaches the value $S = -1$.

g	$\Delta - 4$	g	$\Delta - 4$	g	$\Delta - 4$
0.23	0.53391(8)	0.45	1.4940(6)	0.71	2.5700(5)
0.24	0.57395(9)	0.52	1.7987(8)	0.73	2.6464(1)
0.25	0.61472(0)	0.53	1.8414(9)	0.75	2.7219(2)
0.33	0.95885(9)	0.54	1.8839(8)	0.83	3.016(0)
0.34	1.0032(8)	0.55	1.9262(1)	0.91	3.2988(2)
0.35	1.0478(4)	0.61	2.174(7)	0.93	3.3679(9)
0.43	1.4053(9)	0.63	2.2556(2)	0.96	3.4707(7)
0.44	1.4497(9)	0.65	2.3356(2)	0.98	3.538(6)

Figure 1: Numeric data provided by N.Gromov [78] for the anomalous dimension of the Konishi operator at various values of the coupling g . #

approximant of a function usually has a larger radius of convergency than its Taylor series. And indeed, we empirically observe that the following Pad approximant

$$\Delta_{\text{Konishi}} = 4 + \frac{12g^2 + \sum_{k=1}^{\Lambda} c_k g^{2k}}{1 + \sum_{k=1}^{\Lambda} d_k g^{2k}} \quad (65)$$

converges, when increasing Λ , at least for $g < 0.6$. We took $\Lambda = 10$, expressed the coefficients c_1, c_2, \dots, c_6 through other coefficients so as to reproduce exactly the power series expansion up to seven loops, and then we fitted the the remaining c_i -s and d_i -s against the numerical data for $g < 0.6$. With this procedure, we got a prediction 0.01790×4^{16} for the eight-loop coefficient of the series expansion, which is in perfect agreement with our analytical result (63)^{17,18}.

¹⁷The validity of this approach can be tested by using it to compute 6- and 7-loop quantities first.

¹⁸In fact, one computes rather $\Delta_{\text{Konishi}} - \Delta_{\text{as}}(\tilde{u}_1)$ than Δ_{Konishi} . The magnitude of this difference is about 13% of Δ_{Konishi} , hence the non-triviality of our test is in reproducing the first two digits of the difference.

5. Expansion in inverse transcendentality

A good illustration of the approach explained in section 3 is to perform an expansion in inverse powers of transcendentality. Though this expansion is technically more complicated, there are number of benefits as well. In particular, the less number of iterations is needed to capture interesting higher-wrapping effects. There is also no need to distinguish the ordinary and double scaling regimes, and for the few leading orders of expansion, only quite little should be known about the position of Bethe roots.

The strategy of this expansion is the following. We introduce a variable τ , which will be a bookkeeping variable for the inverse transcendentality order (the transcendentality deficit). We assign this variable to the various quantities according to the following rule

$$\begin{aligned}
g^a &\rightsquigarrow \tau^a g^a, \\
\eta_{a_1, \dots, a_k}^{[r]} &\rightsquigarrow \tau^{-\sum_i a_i} \eta_{a_1, \dots, a_k}^{[r]}, \\
\zeta_{a_1, \dots, a_k} &\rightsquigarrow \tau^{-\sum_i a_i} \zeta_{a_1, \dots, a_k}, \\
\hat{x}, x, z &\rightsquigarrow \hat{x}, x, z, \\
(u+r)^{-a}, \quad a > 0 &\rightsquigarrow \tau^{-a} (u+r)^{-a}.
\end{aligned} \tag{66}$$

After τ is assigned, we perform an expansion in powers of τ . In most cases τ can be treated as an ordinary variable. However, there are exceptions: for instance, when one multiplies $(u+r_1)^{-a_1}$ and $(u+r_2)^{-a_2}$ with $r_1 \neq r_2$, one should perform a partial fraction decomposition (i.e. rewrite the product as a sum of terms with a single pole), giving rise to a sum of terms with different orders in τ , in which the leading term is of order $\tau^{-\max(a_1, a_2)}$. The transcendentality of u in a numerator depends on whether it cancels or not some poles. For us, the latter issue is relevant only for $q_2 = -i u + \dots$, and in this case one can assign τ^{+1} for u . Another example of such exceptions is that the integration parallel to the real axis may increase the leading power of τ by 1 and to produce an expression with mixed powers of τ .

To compute the leading transcendentality coefficients in the anomalous dimension to all orders in g , let us assume that the leading transcendentality q -functions can be derived solely from the large volume solution. This assumption is justified by a careful analysis of equations which can be found in `Transcendentality.nb` [1].

Under this assumption, we get

$$q_{12} = Q + \mathcal{O}(\tau^1), \quad q_2 = -\frac{i}{3g x} \tau^{-1} + \mathcal{O}(\tau^0), \quad U^2 = -\frac{2g^2}{x^2} \tau^2 + \mathcal{O}(\tau^3). \tag{67}$$

The leading transcendentality term for q_{13} is then found from

$$q_{13} = -2g^2\tau^2 Q \sum_{n=0}^{\infty} \left(\frac{1}{x^2 Q^+ Q^-} \right)^{[2n+1]} + \mathcal{O}(\tau^3). \quad (68)$$

In this sum, the highest transcendentality term is determined as follows: at each order of g , we write $\frac{1}{x^2 Q^+ Q^-}$ as a sum of poles $\sum_k \frac{\alpha_k}{(u-v_k)^{n_k}}$, so that the sum over n simplifies to $\sum_k \eta_{n_k}^+(u-v_k)$, where the most transcendental term comes from the pole with the largest exponent. This is the pole coming from $\frac{1}{x^2}$, and its prefactor is the value of $\frac{1}{Q^+ Q^-}$ at $u=0$, namely $9 + \mathcal{O}(g^2)$. Here it was enough to know the leading term in the position of the Bethe root $u_1 = \frac{1}{\sqrt{12}} + \dots$, because the next terms are suppressed in g and hence in τ . We see that we need to know quite little about the position of Bethe roots in order to compute the leading transcendentality contributions.

Let us define, in analogy with (10)¹⁹,

$$\hat{\eta}_{a_1, a_2, \dots, a_k} \equiv \sum_{0 \leq n_1 < n_2 < \dots < n_k < \infty} \frac{1}{(x^{[2n_1]})^{a_1} (x^{[2n_2]})^{a_2} \dots (x^{[2n_k]})^{a_k}}. \quad (69)$$

Note that $\hat{\eta}_I$ has the prefactor τ^0 . For q_{ij} one gets then

$$q_{13} = -18 g^2 \tau^2 Q \hat{\eta}_2^+, \quad q_{14} = 6 i g \tau Q \hat{\eta}_2^+, \quad q_{24} = 2 Q \hat{\eta}_4^+. \quad (70)$$

When we expand H in orders of τ , we encounter the following two combinations:

$$s_1 = \frac{\bar{U}^2}{Q^+} (q_{13}^+ \bar{q}_2^2 + 2q_{14}^+ \bar{q}_2 + q_{24}^+), \quad s_2 = \frac{q_{13}^+ \bar{q}_{24} + 2q_{14}^+ \bar{q}_{14} + q_{24}^+ \bar{q}_{13}}{Q^+ Q^-}, \quad (71)$$

with $s_1 \sim \tau^2$ and $s_2 \sim \tau^2$. The expansion of H in orders of τ gives

$$H = \tau^2 \frac{s_1}{Q^- Q^+} + \frac{\tau^4}{2} \left((9s_1)^2 + 18 s_1 s_2 + 18 \bar{U}^2 \frac{(q_{1,4}^+)^2 - q_{1,3}^+ q_{1,4}^+}{(Q^+)^2} \frac{\bar{q}_{13}^- \bar{q}_2^2 - 2 \bar{q}_{14}^- \bar{q}_2 + \bar{q}_{24}^-}{Q^-} \right) + \mathcal{O}(\tau^5), \quad (72)$$

¹⁹The marginally divergent quantities are defined by $\hat{\eta}_1 \equiv \sum (\frac{1}{\hat{x}^{[2n]}} - \frac{g}{u+i n}) + \eta_1$ and then by stuffle relations.

where the second term is a purely double wrapping effect. We see that at the leading order in τ , the energy is given by the single wrapping term.

The above analysis allows computing the highest transcendentality term to all orders in g . It is coming solely from Δ_{wrap} in (60) and is given by the following explicit integral:

$$\Delta_{\text{lead tran}} = 2i \times (-36 g^2) \int_{-\infty-i0}^{+\infty-i0} \frac{du}{-2\pi i} \frac{1}{\sqrt{1 - \frac{4g^2}{u^2}}} \left(\frac{\hat{\eta}_2^{[2]}}{\hat{x}^4} - \frac{2\hat{\eta}_3^{[2]}}{\hat{x}^3} + \frac{\hat{\eta}_4^{[2]}}{\hat{x}^2} \right). \quad (73)$$

To this end, we can use the Laplace representations

$$\frac{1}{\sqrt{1 - \frac{4g^2}{u^2}}} \frac{1}{\hat{x}^a} = i^a \int_0^\infty dt e^{-i u t} \partial_t J_a(2 g t), \quad (74)$$

$$\hat{\eta}_a^{[2]} = a (-i)^a \int_0^\infty \frac{dt e^{i u t}}{e^t - 1} \frac{J_a(2 g t)}{t}, \quad (75)$$

which are simultaneously valid for $0 > \text{Im}[u] > -1$. After some algebra, one gets

$$\Delta_{\text{lead tran}} = -432 g^2 \int_0^\infty \frac{dt}{e^t - 1} \partial_t \left(\frac{J_3(2 g t)^2}{t} \right). \quad (76)$$

In `Tanscendentality.nb` [1] we also performed a partial analysis for subleading transcendentality terms, in particular for those coming from the double wrapping term in (72), and confirmed the coefficient of the $\zeta_{1,2,8}$ term in the anomalous dimension at eight loops.

6. Conclusions and discussion

In this paper we solved the AdS/CFT Y-system for the Konishi state analytically at weak coupling up to the order where double wrapping effects first appear. This allowed us to compute the Konishi anomalous dimension up to eight loops (63). At this order we observed the appearance of a non-reducible Euler-Zagier sum. In section 3.6 we give a well justified prediction that the answer will be a linear combination of Euler-Zagier sums at any order of perturbation theory. In comparison, a superficial analysis of Feynman graphs allows the appearance of non-reducible Euler-Zagier sums [68], but it

also allows other types of numbers which may appear in the answer starting from nine loops²⁰ [79].

Our computation is based on the FiNLIE that was proposed in [54] and was first adjusted for the weak coupling expansion in [27]. We presented all the integral equations which allow one to straightforwardly perform the weak coupling expansion to any desired order. By contrast, the approach based on Luscher formulae is inapplicable beyond single-wrapping orders²¹.

It appears that at any order of the perturbative expansion, the FiNLIE quantities are expressed as linear combinations of products of a multiple Hurwitz zeta function times the complex conjugate of a multiple Hurwitz zeta function, where the coefficients of the linear combinations are rational functions. We showed that all the integrals can be evaluated analytically in terms of this basis. We developed *Mathematica* packages that handle the technical details of the computation and which were used to compute the leading double wrapping order. It would be possible to apply the FiNLIE approach for other weak coupling computations, e.g for the angle-dependent cusp anomalous dimension (c.f. [40]) or the BMN vacuum of the γ -deformed theory. We also believe that the *Mathematica* tools which we provide here will be useful in a broader context.

The main obstacle for even higher-loop computations is combinatorial growth of the basis of multiple Hurwitz zeta functions which leads to an exponentially increasing demand of computing resources. We estimate, however, that it is technically feasible to reach triple-wrapping orders if a motivation arises for this feat.

An interesting observation is that the algebra of multiple Hurwitz zeta functions appearing in the AdS/CFT integrability respects the stuffle relations, while the algebra of polylogarithms, more typical for perturbative quantum field theory, respects the shuffle relations. Hence, in a sense the two algebras are complementary which might be a hint for a richer Hopf algebra structure of the planar $\mathcal{N}=4$ SYM.

Instead of performing an expansion in the coupling constant, it is also possible

²⁰We thank Matthias Staudacher for clarifications about this point.

²¹For vacuum state the double-wrapping Luscher formulae are known [34]. It is not clear, however, how to generalize these formulae to excited states.

to perform an expansion in inverse orders of transcendentality. This approach captures some effects from higher loops, and in particular we computed the most transcendental terms to all loop orders (76) in section 5. Probably the higher transcendentality terms may also be captured by perturbative quantum field theory methods, which would provide an interesting venue for comparison.

There is another interesting physical phenomenon discussed in section 3.6: The exact Bethe equations can be interpreted as a regularity condition on the Y-system. This feature was already questioned in [54], and its first confirmation was obtained in [27], but only for asymptotic quantities. Here we demonstrated that the cancellation of poles in the q -functions is ensured even if the leading wrapping correction is taken into account. Quite interestingly, we observed a cancellation of poles at the shifted zeroes of q_{12} and not at the shifted positions of the Bethe roots $\pm u_1$. q_{12} is equal to the Baxter polynomial Q only asymptotically, but wrapping effects make it deviate from Q , and then its zeroes acquire a positive imaginary part. On the one hand, $q_{12} \neq Q$ could be an artefact of the computation scheme. On the other hand, Bethe roots are known to become complex in e.g. Lie-Yang model at finite volume [80], hence one can speculate on another interesting possibility: the zeroes q_{12} are an alternative way to parameterize the physical state. This is especially appealing because if we assume that these zeroes are determined by the regularity conditions, then equation (58) should be exactly satisfied. On the other hand, this equation can be equivalently rewritten as an equation

$$\frac{\hat{q}_{123}^{[+1]} \hat{q}_1^{[+1]}}{\hat{q}_{123}^{[-1]} \hat{q}_1^{[-1]}} = -\frac{\hat{q}_{12}^{[+2]}}{\hat{q}_{12}^{[-2]}} \quad \text{for } u - \text{zero of } \hat{q}_{12}, \quad (77)$$

which takes precisely the same form as the Bethe equations appearing in the algebraic Bethe Ansatz solution of spin chains. Hence the AdS/CFT integrable system has one more common feature with integrable systems solvable by an algebraic Bethe Ansatz, in addition to the group-theoretical interpretation of T-functions proposed in [54].

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Appendix A. Further details about η -functions

Appendix A.1. Marginally divergent η -functions

Like in the case of MZVs, the sum (10) is convergent under the conditions $a_k > 1$, $a_k + a_{k-1} > 2$, \dots , $\sum_{i=1}^k a_i > k$. By contrast, in the case of η_1 , the sum is logarithmically divergent, and we *define* the regularized value of η_1 by (12). We then *define* the regularized value of ζ_1 by (11), and this definition gives $\zeta_1 \equiv \gamma_{\text{Euler-Mascheroni}}$.

All the marginally divergent η -functions have the form $\eta_{a_1, \dots, a_k, 1, 1, \dots, 1}$, with $a_k > 1$ or $k = 0$, and we *define* them by requiring that stuffle algebra relations are satisfied. One can check that this definition is self-consistent.

Explicitly, we have

$$\begin{aligned} \eta_{a_1, \dots, a_k, 1} &\equiv \eta_1 \eta_{a_1, \dots, a_k} - \sum_{i=1}^k \eta_{a_1, \dots, a_{i-1}, 1, a_i, \dots, a_k} - \sum_{i=1}^k \eta_{a_1, \dots, a_{i-1}, a_i+1, a_{i+1}, \dots, a_k}, \\ \eta_{a_1, \dots, a_k, 1, 1} &\equiv \frac{1}{2} \left(\eta_1 \eta_{a_1, \dots, a_k, 1} - \sum_{i=1}^k \eta_{a_1, \dots, a_{i-1}, 1, a_i, \dots, a_k, 1} \right. \\ &\quad \left. - \sum_{i=1}^k \eta_{a_1, \dots, a_{i-1}, a_i+1, a_{i+1}, \dots, a_k, 1} - \eta_{a_1, \dots, a_k, 2} \right), \\ &\dots \end{aligned} \tag{A.1}$$

In this way, we recursively show that all marginally divergent functions are expressed as polynomials in η_1 with coefficients which are linear in convergent η -functions. In particular, there is an explicit formula

$$\underbrace{\eta_{1, 1, \dots, 1}}_{(k \text{ times})} = \frac{1}{k!} (\eta_1 + \partial_u)^k \eta_1. \tag{A.2}$$

Using these relations and (11), one defines marginally divergent MZVs.

Appendix A.2. Factorization property

As discussed in section 2.3, there is a simple recursive way to express integrals of the form (14). After a few iterations, any such integral is expressed in terms of standard η -functions evaluated either at point $u = i$ (where they are equal to MZVs (11)) or at point $u = v$, and of generalized η -functions

of the form $\eta_K^{\{\dots, 0, 0, v, 0, 0, \dots\}}$, evaluated at point $u = i$. For instance, for the integral (15), this procedure gives

$$\int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \bar{\eta}_3^{[-2]} \frac{1}{(u-v)} \eta_2^{[2]} = -3\eta_{1,4}^{\{v,0\}} - 2\eta_{2,3}^{\{v,0\}} - \eta_{3,2}^{\{v,0\}} \Big|_{u=i} + \bar{\eta}_3^{[-2]} \eta_2^{[2]} \Big|_{u=v}. \quad (\text{A.3})$$

In the expression obtained by this method, it is quite important that the generalized η -functions that appear have only one non-trivial shift v . This actually allows us to express them in terms of standard η -functions and $\bar{\eta}$ -functions. For example, one obtains #

$$\eta_{2,2}^{\{v,0\}} \Big|_{u=i} = \frac{\zeta_2}{u^2} + 2i\zeta_1 \bar{\eta}_3 - \bar{\eta}_{2,2} - 2\bar{\eta}_{3,1} \Big|_{u=v}. \quad (\text{A.4})$$

The proof of this factorization property is done by induction over the depth of the η -functions. First, it is clear that $\eta_a^{\{v\}} \Big|_{u=i} = \eta_a(i-v) = (-1)^a \bar{\eta}_a^{[-2]} \Big|_{u=v}$. Let us now assume that, when the depth is smaller than n , we know how to express $\eta_K^{\{\dots, 0, 0, v, 0, 0, \dots\}} \Big|_{u=i}$ in terms of standard η -functions and $\bar{\eta}$ -functions. Then, for an arbitrary multi-index $I = a, \tilde{I}$ with n elements, we have two different ways to compute the integral

$$\int_{-\infty}^{+\infty} \frac{du}{-2\pi i} \frac{\bar{\eta}_{\tilde{I},1}^{[-2]}}{(u-v)^a} \quad (\text{A.5})$$

for $0 < \text{Im}(v) < 1$: either we close the integration contour downwards, and we immediately obtain that this integral is zero. Or we close it upwards, using the iterative procedure from above. Except for the very last residue, at each step we will obtain expressions involving η -functions with generalized shifts of the form $\eta_K^{\{\dots, 0, 0, v, 0, 0, \dots\}}$, where K has less than n elements. On the other hand, if we denote $\tilde{I} = a_n, a_{n-1}, \dots, a_1$ (where $I = a_1, a_2, \dots, a_n$), then the residue at the very last step is $\text{Res}_{u=0} \frac{(\bar{\eta}_{\tilde{I}}^{\{\dots, 0, 0, v\}})^{[+2]}}{u} = \bar{\eta}_{\tilde{I}}^{\{\dots, 0, 0, v\}} \Big|_{u=i}$. Hence the vanishing of the integral (A.5) allows us to express $\bar{\eta}_{\tilde{I}}^{\{\dots, 0, 0, v\}} \Big|_{u=i}$ in terms of usual η -functions. For instance, this argument allows to derive the relation

$$\bar{\eta}_{2,2}^{\{0,v\}} \Big|_{u=i} = -\zeta_2 \eta_2^{[2]} + 2i\zeta_1 \eta_3^{[2]} + \eta_{2,2}^{[2]} + 2\eta_{3,1}^{[2]} \Big|_{u=v}. \quad (\text{A.6})$$

Finally, one notes that the functions $\eta_K^{\{\dots, 0, 0, v, 0, 0, \dots\}}$ where the shift v is not in the last position are expressed through the functions $\eta_{\tilde{K}}^{\{\dots, 0, 0, v\}}$ using the stuffle algebra. For instance, we have

$$\eta_{a,b,c}^{\{0,v,0\}} = \eta_{a,b}^{\{0,v\}} \eta_c - \eta_{a,c,b}^{\{0,0,v\}} - \eta_{c,a,b}^{\{0,0,v\}} + \dots, \quad (\text{A.7})$$

where the period (\dots) denotes η -functions with less indices.

Hence, we showed how to express any basic integral (14) in terms of η - and $\bar{\eta}$ -functions, MZVs, and rational functions of u .

It is clear that if $\text{Im}(v) > 0$, the answer for the integral (14) should be analytic in the upper half-plane. We will now describe relations that allow canceling all the $\bar{\eta}$ -functions using the periodicity property, and these relations can be used to make the analyticity of integrals of the form (14) manifest.

Appendix A.3. Periodicity property

There is a certain class of relations between $\bar{\eta}$ - and η -functions. One example of such relation is:

$$\left(\eta_2^{[+2]} + \bar{\eta}_2\right)^2 = -4\zeta_2 \left(\eta_2^{[+2]} + \bar{\eta}_2\right) + \left(\eta_4^{[+2]} + \bar{\eta}_4\right). \quad (\text{A.8})$$

To prove the relation (A.8), we notice that the function $(\eta_2^{[+2]} + \bar{\eta}_2)^2 = \left(\sum_{k \in \mathbb{Z}} \frac{1}{(u+ik)^2}\right)^2$ is a periodic function with period i , which has the Laurent series expansion $(\eta_2^{[+2]} + \bar{\eta}_2)^2 = \frac{1}{u^4} - \frac{4\zeta_2}{u^2} + \frac{44}{5}\zeta_2^2 + \mathcal{O}(u^2)$ in the vicinity of zero, which is its only singularity lying in the strip $\{u : |\text{Im}(u)| \leq 1/2\}$. In the r.h.s. of (A.8), the coefficient $-4\zeta_2$ is chosen in such a way that the r.h.s., which is also periodic with poles on $i\mathbb{Z}$, has the same Laurent series in the vicinity of zero. Hence, since also both the l.h.s. and the r.h.s. tend to zero when $u \rightarrow \pm\infty$, they should be equal.

Using the same idea, we will actually show that an arbitrary function $\bar{\eta}_I$ can be expressed in terms of the functions η_J and of the function $\bar{\eta}_1$. To this end, let us study the periodic function

$$\mathcal{P}_{a,I} = \sum_{k \in \mathbb{Z}} \frac{1}{(u+ik)^a} \eta_I(u + i(k+1)). \quad (\text{A.9})$$

If I is the empty multi-index, we use the convention $\eta_\emptyset = 1$, hence we have

$$\mathcal{P}_a = \sum_{k \in \mathbb{Z}} \frac{1}{(u+ik)^a} = \eta_a^{[+2]} + \bar{\eta}_a. \quad (\text{A.10})$$

Let us denote by i_2, i_3, \dots, i_d the elements of I , and define $i_1 = a$. Then one can generalize the relation (A.10) as follows:

$$\mathcal{P}_{a,I} = \sum_{n_1 < n_2 < \dots < n_d} \left(\frac{1}{u^{i_1}} \right)^{[2n_1]} \left(\frac{1}{u^{i_2}} \right)^{[2n_2]} \dots \left(\frac{1}{u^{i_d}} \right)^{[2n_d]} \quad (\text{A.11})$$

$$= \sum_{k=0}^d \sum_{n_1 < n_2 < \dots < n_k \leq 0 < n_{k+1} < \dots < n_d} \prod_{p=1}^d \left(\frac{1}{u^{i_p}} \right)^{[2n_p]} \quad (\text{A.12})$$

$$= \sum_{k=0}^d \eta_{i_{k+1}, i_{k+2}, \dots, i_d}^{[2]} \bar{\eta}_{i_k, i_{k-1}, \dots, i_1}, \quad (\text{A.13})$$

where we remind the convention $\eta_\emptyset = \bar{\eta}_\emptyset = 1$. For instance, if I has two elements b and c , then we have

$$\mathcal{P}_{a,b,c} = \eta_{a,b,c}^{[2]} + \eta_{b,c}^{[2]} \bar{\eta}_a + \eta_c^{[2]} \bar{\eta}_{b,a} + \bar{\eta}_{c,b,a}. \quad (\text{A.14})$$

Using the periodicity of \mathcal{P}_I , we can then find a set of constants γ_k such that $\mathcal{P}_I = \sum_k \gamma_k \left(\eta_1^{[+2]} + \bar{\eta}_1 \right)^k$: these constants are found by requiring that the singular and constant parts of the Laurent series of the r.h.s. and the l.h.s. do match. Using the expression (A.13) of \mathcal{P}_I , this allows writing $\bar{\eta}_I = \sum_k \gamma_k \left(\eta_1^{[+2]} + \bar{\eta}_1 \right)^k - \sum_{k=0}^{d-1} \eta_{i_{k+1}, i_{k+2}, \dots, i_d}^{[2]} \bar{\eta}_{i_k, i_{k-2}, \dots, i_1}$. We can therefore iteratively express an arbitrary function $\bar{\eta}_I$ in terms of functions η_J and of the function $\bar{\eta}_1$.

References

- [1] S. Leurent, D. Volin, *Mathematica* packages for working with zeta functions and FiNLIE-based weak coupling expansion.
URL <http://people.kth.se/~dmytrov/konishi8.zip>
- [2] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, et al., Review of AdS/CFT Integrability: An Overview, *Lett.Math.Phys.* 99 (2012) 3–32. [arXiv:1012.3982](#), doi:10.1007/s11005-011-0529-2.
- [3] J. Ambjorn, R. A. Janik, C. Kristjansen, Wrapping interactions and a new source of corrections to the spin-chain / string duality, *Nucl. Phys. B* 736 (2006) 288–301. [arXiv:hep-th/0510171](#), doi:10.1016/j.nuclphysb.2005.12.007.

- [4] N. Beisert, M. Staudacher, Long-range $PSU(2, 2|4)$ Bethe Ansatz for gauge theory and strings, Nucl. Phys. B727 (2005) 1–62. [arXiv:hep-th/0504190](#), [doi:10.1016/j.nuclphysb.2005.06.038](#).
- [5] N. Beisert, B. Eden, M. Staudacher, Transcendentality and Crossing, J.Stat.Mech. 0701 (2007) P01021. [arXiv:hep-th/0610251](#), [doi:10.1088/1742-5468/2007/01/P01021](#).
- [6] N. Gromov, V. Kazakov, P. Vieira, Exact Spectrum of Anomalous Dimensions of Planar $N=4$ Supersymmetric Yang-Mills Theory, Phys. Rev. Lett. 103 (2009) 131601. [arXiv:0901.3753](#), [doi:10.1103/PhysRevLett.103.131601](#).
- [7] N. Gromov, V. Kazakov, P. Vieira, Exact Spectrum of Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory: Konishi Dimension at Any Coupling, Phys. Rev. Lett. 104 (2010) 211601. [arXiv:0906.4240](#), [doi:10.1103/PhysRevLett.104.211601](#).
- [8] S. Frolov, Konishi operator at intermediate coupling, J. Phys. A44 (2011) 065401. [arXiv:1006.5032](#), [doi:10.1088/1751-8113/44/6/065401](#).
- [9] S. Frolov, Scaling dimensions from the mirror TBA, J.Phys. A45 (2012) 305402. [arXiv:1201.2317](#), [doi:10.1088/1751-8113/45/30/305402](#).
- [10] N. Gromov, D. Serban, I. Shenderovich, D. Volin, Quantum folded string and integrability: From finite size effects to Konishi dimension, JHEP 1108 (2011) 046. [arXiv:1102.1040](#), [doi:10.1007/JHEP08\(2011\)046](#).
- [11] N. Gromov, S. Valatka, Deeper Look into Short Strings, JHEP 1203 (2012) 058. [arXiv:1109.6305](#), [doi:10.1007/JHEP03\(2012\)058](#).
- [12] R. Roiban, A. A. Tseytlin, Quantum strings in $AdS_5 \times S^5$: strong-coupling corrections to dimension of Konishi operator, JHEP 11 (2009) 013. [arXiv:0906.4294](#), [doi:10.1088/1126-6708/2009/11/013](#).
- [13] R. Roiban, A. Tseytlin, Semiclassical string computation of strong-coupling corrections to dimensions of operators in Konishi multiplet, Nucl.Phys. B848 (2011) 251–267. [arXiv:1102.1209](#), [doi:10.1016/j.nuclphysb.2011.02.016](#).

- [14] B. C. Vallilo, L. Mazzucato, The Konishi multiplet at strong coupling, JHEP 1112 (2011) 029. [arXiv:1102.1219](#), [doi:10.1007/JHEP12\(2011\)029](#).
- [15] Z. Bajnok, R. A. Janik, Four-loop perturbative Konishi from strings and finite size effects for multiparticle states, Nucl. Phys. B807 (2009) 625–650. [arXiv:0807.0399](#), [doi:10.1016/j.nuclphysb.2008.08.020](#).
- [16] Z. Bajnok, A. Hegedus, R. A. Janik, T. Lukowski, Five loop Konishi from AdS/CFT, Nucl.Phys. B827 (2010) 426–456. [arXiv:0906.4062](#), [doi:10.1016/j.nuclphysb.2009.10.015](#).
- [17] G. Arutyunov, S. Frolov, R. Suzuki, Five-loop Konishi from the Mirror TBA, JHEP 04 (2010) 069. [arXiv:1002.1711](#), [doi:10.1007/JHEP04\(2010\)069](#).
- [18] J. Balog, A. Hegedus, 5-loop Konishi from linearized TBA and the XXX magnet, JHEP 06 (2010) 080. [arXiv:1002.4142](#), [doi:10.1007/JHEP06\(2010\)080](#).
- [19] F. Fiamberti, A. Santambrogio, C. Sieg, D. Zanon, Wrapping at four loops in N=4 SYM, Phys.Lett. B666 (2008) 100–105. [arXiv:0712.3522](#), [doi:10.1016/j.physletb.2008.06.061](#).
- [20] F. Fiamberti, A. Santambrogio, C. Sieg, D. Zanon, Anomalous dimension with wrapping at four loops in N=4 SYM, Nucl.Phys. B805 (2008) 231–266. [arXiv:0806.2095](#), [doi:10.1016/j.nuclphysb.2008.07.014](#).
- [21] V. Velizhanin, The four-loop anomalous dimension of the Konishi operator in N=4 supersymmetric Yang-Mills theory, JETP Lett. 89 (2009) 6–9. [arXiv:0808.3832](#), [doi:10.1134/S0021364009010020](#).
- [22] B. Eden, P. Heslop, G. P. Korchemsky, V. A. Smirnov, E. Sokatchev, Five-loop Konishi in N=4 SYM, Nucl.Phys. B862 (2012) 123–166. [arXiv:1202.5733](#), [doi:10.1016/j.nuclphysb.2012.04.015](#).
- [23] Z. Bajnok, R. A. Janik, T. Lukowski, Four loop twist two, BFKL, wrapping and strings, Nucl. Phys. B816 (2009) 376–398. [arXiv:0811.4448](#), [doi:10.1016/j.nuclphysb.2009.02.005](#).

- [24] T. Lukowski, A. Rej, V. N. Velizhanin, Five-Loop Anomalous Dimension of Twist-Two Operators, Nucl. Phys. B831 (2010) 105–132. [arXiv:0912.1624](#), [doi:10.1016/j.nuclphysb.2010.01.008](#).
- [25] J. Balog, A. Hegedus, The Bajnok-Janik formula and wrapping corrections, JHEP 09 (2010) 107. [arXiv:1003.4303](#), [doi:10.1007/JHEP09\(2010\)107](#).
- [26] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher, V. N. Velizhanin, Dressing and Wrapping, J. Stat. Mech. 0710 (2007) P10003. [arXiv:0704.3586](#), [doi:10.1088/1742-5468/2007/10/P10003](#).
- [27] S. Leurent, D. Serban, D. Volin, Six-Loop Konishi Anomalous Dimension from the Y System, Phys. Rev. Lett. 109 (2012) 241601. [arXiv:1209.0749](#), [doi:10.1103/PhysRevLett.109.241601](#).
- [28] Z. Bajnok, R. A. Janik, Six and seven loop Konishi from Luscher corrections, JHEP 1211 (2012) 002. [arXiv:1209.0791](#), [doi:10.1007/JHEP11\(2012\)002](#).
- [29] G. Arutyunov, S. Frolov, S. J. van Tongeren, Bound States in the Mirror TBA, JHEP 1202 (2012) 014. [arXiv:1111.0564](#), [doi:10.1007/JHEP02\(2012\)014](#).
- [30] G. Arutyunov, S. Frolov, A. Sfondrini, Exceptional Operators in $N=4$ super Yang-Mills, JHEP 1209 (2012) 006. [arXiv:1205.6660](#), [doi:10.1007/JHEP09\(2012\)006](#).
- [31] N. Gromov, F. Levkovich-Maslyuk, Y -system and β -deformed $N=4$ Super-Yang-Mills, J.Phys. A44 (2011) 015402. [arXiv:1006.5438](#), [doi:10.1088/1751-8113/44/1/015402](#).
- [32] G. Arutyunov, M. de Leeuw, S. J. van Tongeren, Twisting the Mirror TBA, JHEP 1102 (2011) 025. [arXiv:1009.4118](#), [doi:10.1007/JHEP02\(2011\)025](#).
- [33] Z. Bajnok, O. el Deeb, 6-loop anomalous dimension of a single impurity operator from AdS/CFT and multiple zeta values, JHEP 1101 (2011) 054. [arXiv:1010.5606](#), [doi:10.1007/JHEP01\(2011\)054](#).

- [34] C. Ahn, Z. Bajnok, D. Bombardelli, R. I. Nepomechie, TBA, NLO Luscher correction, and double wrapping in twisted AdS/CFT, JHEP 1112 (2011) 059. [arXiv:1108.4914](#), [doi:10.1007/JHEP12\(2011\)059](#).
- [35] M. de Leeuw, S. J. van Tongeren, The spectral problem for strings on twisted $\text{AdS}_5 \times \text{S}^5$, Nucl.Phys. B860 (2012) 339–376. [arXiv:1201.1451](#), [doi:10.1016/j.nuclphysb.2012.03.004](#).
- [36] G. Arutyunov, M. de Leeuw, S. J. van Tongeren, The Quantum Deformed Mirror TBA I, JHEP 1210 (2012) 090. [arXiv:1208.3478](#), [doi:10.1007/JHEP10\(2012\)090](#).
- [37] G. Arutyunov, M. de Leeuw, S. J. van Tongeren, The Quantum Deformed Mirror TBA II, JHEP 1302 (2013) 012. [arXiv:1210.8185](#), [doi:10.1007/JHEP02\(2013\)012](#).
- [38] D. Correa, J. Maldacena, A. Sever, The quark anti-quark potential and the cusp anomalous dimension from a TBA equation, JHEP 1208 (2012) 134. [arXiv:1203.1913](#), [doi:10.1007/JHEP08\(2012\)134](#).
- [39] N. Drukker, Integrable Wilson loops, [arXiv:1203.1617](#).
- [40] N. Gromov, A. Sever, Analytic Solution of Bremsstrahlung TBA, JHEP 1211 (2012) 075. [arXiv:1207.5489](#), [doi:10.1007/JHEP11\(2012\)075](#).
- [41] B. Basso, An exact slope for AdS/CFT, [arXiv:1109.3154](#).
- [42] P. Y. Casteill, C. Kristjansen, The Strong Coupling Limit of the Scaling Function from the Quantum String Bethe Ansatz, Nucl. Phys. B785 (2007) 1–18. [arXiv:0705.0890](#), [doi:10.1016/j.nuclphysb.2007.06.011](#).
- [43] N. Gromov, Generalized Scaling Function at Strong Coupling, JHEP 11 (2008) 085. [arXiv:0805.4615](#), [doi:10.1088/1126-6708/2008/11/085](#).
- [44] D. Volin, The 2-loop generalized scaling function from the BES/FRS equation, [arXiv:0812.4407](#).
- [45] B. Basso, G. P. Korchemsky, J. Kotanski, Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling, Phys. Rev. Lett. 100 (2008) 091601. [arXiv:0708.3933](#), [doi:10.1103/PhysRevLett.100.091601](#).

- [46] I. Kostov, D. Serban, D. Volin, Functional BES equation, JHEP 08 (2008) 101. [arXiv:0801.2542](#), [doi:10.1088/1126-6708/2008/08/101](#).
- [47] A. Hegedus, Private communication.
- [48] Z. Bajnok, Private communication.
- [49] D. Bombardelli, D. Fioravanti, R. Tateo, Thermodynamic Bethe Ansatz for planar AdS/CFT: A Proposal, J.Phys.A A42 (2009) 375401. [arXiv:0902.3930](#), [doi:10.1088/1751-8113/42/37/375401](#).
- [50] N. Gromov, V. Kazakov, A. Kozak, P. Vieira, Exact Spectrum of Anomalous Dimensions of Planar $N = 4$ Supersymmetric Yang-Mills Theory: TBA and excited states, Lett. Math. Phys. 91 (2010) 265–287. [arXiv:0902.4458](#), [doi:10.1007/s11005-010-0374-8](#).
- [51] G. Arutyunov, S. Frolov, Thermodynamic Bethe Ansatz for the $AdS_5 \times S^5$ Mirror Model, JHEP 05 (2009) 068. [arXiv:0903.0141](#), [doi:10.1088/1126-6708/2009/05/068](#).
- [52] A. Cavaglia, D. Fioravanti, R. Tateo, Extended Y-system for the AdS_5/CFT_4 correspondence, Nucl. Phys. B843 (2011) 302–343. [arXiv:1005.3016](#), [doi:10.1016/j.nuclphysb.2010.09.015](#).
- [53] J. Balog, A. Hegedus, $AdS_5 \times S^5$ mirror TBA equations from Y-system and discontinuity relations, JHEP 1108 (2011) 095. [arXiv:1104.4054](#), [doi:10.1007/JHEP08\(2011\)095](#).
- [54] N. Gromov, V. Kazakov, S. Leurent, D. Volin, Solving the AdS/CFT Y-system, JHEP 1207 (2012) 023. [arXiv:1110.0562](#), [doi:10.1007/JHEP07\(2012\)023](#).
- [55] I. Krichever, O. Lipan, P. Wiegmann, A. Zabrodin, Quantum integrable models and discrete classical Hirota equations, Commun. Math. Phys. 188 (1997) 267–304. [arXiv:hep-th/9604080](#), [doi:10.1007/s002200050165](#).
- [56] N. Gromov, V. Kazakov, Z. Tsuboi, $PSU(2,2|4)$ Character of Quasiclassical AdS/CFT, JHEP 07 (2010) 097. [arXiv:1002.3981](#), [doi:10.1007/JHEP07\(2010\)097](#).

- [57] N. Gromov, V. Kazakov, S. Leurent, Z. Tsuboi, Wronskian Solution for AdS/CFT Y-system, JHEP 1101 (2011) 155. [arXiv:1010.2720](#), [doi:10.1007/JHEP01\(2011\)155](#).
- [58] Z. Tsuboi, Solutions of the T-system and Baxter equations for supersymmetric spin chains, Nucl.Phys. B826 (2010) 399–455. [arXiv:0906.2039](#), [doi:10.1016/j.nuclphysb.2009.08.009](#).
- [59] Z. Tsuboi, Wronskian solutions of the T, Q and Y-systems related to infinite dimensional unitarizable modules of the general linear superalgebra $gl(M|N)$, Nuclear Physics B 870 (1) (2013) 92 – 137. [arXiv:1109.5524](#), [doi:10.1016/j.nuclphysb.2013.01.007](#).
- [60] D. Volin, String hypothesis for $gl(n|m)$ spin chains: a particle/hole democracy, Lett.Math.Phys. 102 (2012) 1–29. [arXiv:1012.3454](#), [doi:10.1007/s11005-012-0570-9](#).
- [61] N. Gromov, V. Kazakov, S. Leurent, D. Volin, Work in progress.
- [62] W. Zudilin, Algebraic relations for multiple zeta values, Uspekhi Mat. Nauk 58 (1) (2003) 3–32. [doi:10.1070/RM2003v058n01ABEH000592](#).
- [63] M. Waldschmidt, Valeurs zêta multiples. Une introduction, Journal de théorie des nombres de Bordeaux 12 (4) (2000) 581–595. [doi:10.5802/jtnb.298](#).
- [64] J. Ablinger, J. Blumlein, C. Schneider, Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms [arXiv:1302.0378](#).
- [65] A. Goncharov, Multiple polylogarithms, cyclotomy and modular complexes, Mathematical Research Letters 5 (1998) 497–516. [arXiv:1105.2076](#).
- [66] M. E. Hoffman, The algebra of multiple harmonic series, Journal of Algebra 194 (2) (1997) 477 – 495. [doi:10.1006/jabr.1997.7127](#).
- [67] D. H. Bailey, D. J. Broadhurst, Parallel integer relation detection: Techniques and applications, Math. Comput. 70 (236) (2001) 1719–1736. [doi:10.1090/S0025-5718-00-01278-3](#).

- [68] D. J. Broadhurst, D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, *Phys.Lett. B* 393 (1997) 403–412. [arXiv:hep-th/9609128](#), [doi:10.1016/S0370-2693\(96\)01623-1](#).
- [69] Y. Ohno, A generalization of the duality and sum formulas on the multiple zeta values, *Journal of Number Theory* 74 (1) (1999) 39 – 43. [doi:10.1006/jnth.1998.2314](#).
- [70] M. E. Hoffman, Y. Ohno, Relations of multiple zeta values and their algebraic expression, *J. Algebra* 262 (2003) 332–347. [arXiv:math.QA/0010140](#).
- [71] K. Ihara, M. Kaneko, D. Zagier, Derivation and double shuffle relations for multiple zeta values, *Compositio Mathematica* 142 307–338. [doi:10.1112/S0010437X0500182X](#).
- [72] J. Blumlein, D. Broadhurst, J. Vermaseren, The Multiple Zeta Value Data Mine, *Comput.Phys.Commun.* 181 (2010) 582–625. [arXiv:0907.2557](#), [doi:10.1016/j.cpc.2009.11.007](#).
- [73] J. Ablinger, J. Blumlein, C. Schneider, Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials, *J.Math.Phys.* 52 (2011) 102301. [arXiv:1105.6063](#), [doi:10.1063/1.3629472](#).
- [74] N. Gromov, V. Kazakov, P. Vieira, Finite volume spectrum of 2D field theories from Hirota dynamics, *JHEP* 0912 (2009) 060. [arXiv:0812.5091](#), [doi:10.1088/1126-6708/2009/12/060](#).
- [75] V. Kazakov, S. Leurent, Finite size spectrum of $SU(N)$ principal chiral field from discrete Hirota dynamics, [arXiv:1007.1770](#).
- [76] R. A. Janik, The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry, *Phys. Rev. D* 73 (2006) 086006. [arXiv:hep-th/0603038](#), [doi:10.1103/PhysRevD.73.086006](#).
- [77] D. Volin, Minimal solution of the AdS/CFT crossing equation, *J.Phys. A* 42 (2009) 372001. [arXiv:0904.4929](#), [doi:10.1088/1751-8113/42/37/372001](#).
- [78] N. Gromov, Private communication.

- [79] F. Brown, O. Schnetz, A K3 in ϕ^4 , Duke Mathematical Journal 161 (10) (2012) 1817–1862. [arXiv:1006.4064](#), [doi:10.1215/00127094-1644201](#).
- [80] P. Dorey, R. Tateo, Excited states by analytic continuation of TBA equations, Nucl. Phys. B482 (1996) 639–659. [arXiv:hep-th/9607167](#), [doi:10.1016/S0550-3213\(96\)00516-0](#).